МИНИСТЕРСТВО ОБРАЗОВАНИЯ И НАУКИ

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В настоящем пособии изложены учебно-методические материалы по курсу “Дифференциальные уравнения” для иностранных студентов, обучающихся в ННГУ по направлению подготовки 010300 «Фундаментальная информатика и информационные технологии» (бакалавриат).

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R.S.Biryukov

Systems of Linear Differential Equations

Study book

Recommended by the Methodical Commission of the Faculty of Computer Science
for international students, studying at the B.Sc. programme “010300”

“Fundamental Informatics and Information Technologies”

Nizhny Novgorod

2013

Introduction

There are many physical problems that involve a number of separate elements linked together in some manner. For example, electrical networks have this character, as do some problems in mechanics or in other fields. In these and similar cases the corresponding mathematical problem consists of a system of two or more differential equations, which can always be written as a first order equations. We consider systems of linear first order differential equations. After describing some of the most basic ideas and terminology, we will look at homogeneous systems with constant coefficients. Then for nonhomogeneous systems we describe the method of undetermined coefficients and the method of variation of parameters.

1. Preliminary Theory

A system of first order linear differential equations is a collection of n interrelated differential equations of the form

$$\begin{cases} \dot{x}_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + b_1(t), \\ \dot{x}_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + b_2(t), \\ \quad\vdots \\ \dot{x}_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + b_n(t), \end{cases} \tag{1.1}$$

where the functions $a_{11}(t), \ldots, a_{nn}(t)$ and $b_1(t), \ldots, b_n(t)$ are real-valued functions defined on the interval $I_{\alpha,\beta} := \{\alpha < t < \beta\}$. If each of the functions $b_1(t), \ldots, b_n(t)$ is zero for all t in the interval $I_{\alpha,\beta}$, then the system (1.1) is said to be *homogeneous*; otherwise, it is *nonhomogeneous*.

The system (1.1) is said to have a *solution* on the interval $I_{\alpha,\beta}$ if there exists a set of n functions

$$x_1(t) = \varphi_1(t), \quad x_2(t) = \varphi_2(t), \quad \ldots, \quad x_n = \varphi_n(t), \tag{1.2}$$

that are differentiable and satisfy the system of Eq. (1.1) at all points in this interval. In addition to the given system of differential equations there may also be given initial conditions of the form

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0, \tag{1.3}$$

where t_0 is a specified value of t in $I_{\alpha,\beta}$, and $x_1^0, x_2^0, \ldots, x_n^0$ are prescribed numbers. The differential equations (1.1) and initial conditions (1.3) together form an initial value problem. The following conditions on coefficients $a_{11}(t), \ldots, a_{nn}(t)$ and $b_1(t), \ldots, b_n(t)$ are sufficient to assure that the initial value problem (1.2), (1.3) has a unique solution.

Theorem 1.1. If the functions $a_{11}(t), \ldots, a_{nn}(t)$ and $b_1(t), \ldots, b_n(t)$ are continuous on an open interval $I_{\alpha,\beta}$, then there exists a unique solution (1.2) of the system (1.1) that also satisfies the initial conditions (1.3), where t_0 is any point in $I_{\alpha,\beta}$ and $x_1^0, \ldots, x_n^0$ are any prescribed numbers. Moreover, the solution exists throughout the interval $I_{\alpha,\beta}$.

To simplify notation, we write system (1.1) in matrix form. That is, let $\boldsymbol{x}(t)$, $\boldsymbol{A}(t)$ and $\boldsymbol{b}(t)$ denote the respective matrices

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \boldsymbol{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix}, \quad \boldsymbol{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix},$$

then the system (1.1) takes the form

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

or simply

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{b}(t). \tag{1.4}$$

A vector $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ is said to be a *solution* of system (1.4) if its components (1.2) satisfy the system of equations (1.1). Throughout this section we assume that $\boldsymbol{A}(t)$ and $\boldsymbol{b}(t)$ are continuous on some interval $I_{\alpha,\beta}$. According to Theorem 1.1, this is sufficient to guarantee the existence of solutions of Eq. (1.4) on the interval $I_{\alpha,\beta}$.

It is convenient to consider first the homogeneous equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}. \tag{1.5}$$

Once the homogeneous equation has been solved, there are two methods that can be used to solve the nonhomogeneous equation (1.4): the method of undetermined coefficients and the method of variation of parameters. The following result is a superposition principle for solutions of linear homogeneous systems.

Theorem 1.2. If the vector functions $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_k(t)$ are solutions of system (1.5), then the linear combination

$$\boldsymbol{x}(t) = C_1\boldsymbol{x}_1(t) + C_2\boldsymbol{x}_2(t) + \ldots + C_k\boldsymbol{x}_k(t)$$

is also a solution for any constants $C_1, C_2, \ldots, C_k$.

Using the superposition principle we are able to find all solutions of Eq. (1.5), it is sufficient to form linear combinations of n properly chosen solutions. Let $\boldsymbol{x}_1(t)$, $\boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)$ be such n solutions of the nth order system (1.5), and consider the matrix $\boldsymbol{X}(t)$ whose columns are the vectors $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)$:

$$\boldsymbol{X}(t) = \begin{pmatrix} x_1^1(t) & x_1^2(t) & \ldots & x_1^n(t) \\ x_2^1(t) & x_2^2(t) & \ldots & x_2^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1(t) & x_n^2(t) & \ldots & x_n^n(t) \end{pmatrix}, \tag{1.6}$$

where $x_i^j(t)$ is the ith component of the jth solution $\boldsymbol{x}_j(t)$. The columns of $\boldsymbol{X}(t)$ are linearly independent, and therefore, solutions $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)$ are linearly independent too, if and only if $\det \boldsymbol{X}(t) \neq 0$. This determinant is denoted by

$$W[\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)] = \det \boldsymbol{X}(t)$$

and called the Wronskian of the n solutions $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)$. Note that if $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)$ are solutions of Eq. (1.5) on the interval $I_{\alpha,\beta}$, then in this interval $W[\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)]$ either is identically zero or else never vanishes.

Theorem 1.3. If the vector functions $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)$ are linearly independent solutions of Eq. (1.5) for each point in the interval $I_{\alpha,\beta}$, then each solution $\boldsymbol{\varphi}(t)$ of this system can be expressed as a linear combination of $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)$,

$$\boldsymbol{\varphi}(t) = C_1\boldsymbol{x}_1(t) + C_2\boldsymbol{x}_2(t) + \ldots + C_n\boldsymbol{x}_n(t), \tag{1.7}$$

in exactly one way.

Note that according to Theorem 1.2 all expressions of the form (1.7) are solutions of the system (1.5), while by Theorem 1.3 all solutions of Eq. (1.5) can be written in the form (1.7). If the constants $C_1, C_2, \ldots, C_n$ are thought of as arbitrary, then Eq. (1.7) includes all solutions of the system (1.5), and it is customary to call it the general solution. Any set of solutions $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)$ of Eq. (1.5), which is linearly independent at each point in the interval $I_{\alpha,\beta}$, is said to be a *fundamental set of solutions* for that interval.

2. Homogeneous Linear Systems with Constant Coefficients

We will concentrate most of our attention on systems of homogeneous linear equations with constant coefficients; that is, systems of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \tag{2.1}$$

where $\boldsymbol{A}$ is a constant $n \times n$ matrix with real-valued coefficients.

To construct the general solution of the system (2.1) we seek it in the form

$$\boldsymbol{x} = \boldsymbol{v}e^{\lambda t}, \tag{2.2}$$

where the exponent λ and the constant vector $\boldsymbol{v}$ are to be determined. Substituting from Eq. (2.2) for $\boldsymbol{x}$ in the system (2.1) gives

$$\lambda \boldsymbol{v}e^{\lambda t} = \boldsymbol{A}\boldsymbol{v}e^{\lambda t}.$$

Upon canceling the nonzero scalar factor $e^{\lambda t}$ we obtain

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = 0, \tag{2.3}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix. Thus, to solve the system of differential equations (2.1), we must solve the system of algebraic equations (2.3). This latter problem is precisely the one that determines the eigenvalues λ and the eigenvectors $\boldsymbol{v}$ of the matrix $\boldsymbol{A}$. The *eigenvalues* λ are roots of the *characteristic equation* of the matrix $\boldsymbol{A}$:

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0. \tag{2.4}$$

A solution $\boldsymbol{v} \neq 0$ of (2.3) corresponding to an eigenvalue $\boldsymbol{A}$ is called an *eigenvector* of $\boldsymbol{A}$. The nature of the eigenvalues and the corresponding eigenvectors determines the nature of the general solution of the system (2.1). Because of assumption that $\boldsymbol{A}$ is a real-valued matrix, there are three possibilities for the eigenvalues of $\boldsymbol{A}$: all eigenvalues are real and different from each other, some eigenvalues occur in complex conjugate pairs, some eigenvalues are repeated.

Distinct Real Eigenvalues. If the eigenvalues are all real and different, then associated with each eigenvalue λ_i is a real eigenvector $\boldsymbol{v}_i$ and the set of n eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is linearly independent. The corresponding solutions of the differential system (2.1) are

$$\boldsymbol{x}_1 = \boldsymbol{v}_1 e^{\lambda_1 t}, \quad \boldsymbol{x}_2 = \boldsymbol{v}_2 e^{\lambda_2 t}, \quad \ldots, \quad \boldsymbol{x}_n = \boldsymbol{v}_n e^{\lambda_n t}. \tag{2.5}$$

To show that these solutions form a fundamental set, we evaluate their Wronskian:

$$W[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n](t) = \det(\boldsymbol{v}_1 e^{\lambda_1 t}, \ldots, \boldsymbol{v}_n e^{\lambda_n t}) = e^{(\lambda_1 + \ldots + \lambda_n)t} \det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n).$$

First, we observe that the exponential function is never zero. Next, since the eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent, the determinant in the last term is nonzero. As a consequence, the Wronskian $W[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n](t)$ is never zero; hence $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ form a fundamental set of solutions. Thus the general solution of Eq. (2.1) is

$$\boldsymbol{x} = C_1 \boldsymbol{v}_1 e^{\lambda_1 t} + C_2 \boldsymbol{v}_2 e^{\lambda_2 t} + \ldots + C_n \boldsymbol{v}_n e^{\lambda_n t}. \tag{2.6}$$

Example 2.1. Find the general solution of the following system:

$$\begin{cases} \dot{x} = 2x + 3y, \\ \dot{y} = 2x + y. \end{cases}$$

To find the general solution of the system, we first have to find a fundamental set of solutions. For this we assume that $\boldsymbol{x} = \boldsymbol{v}e^{\lambda t}$ and obtain the set of linear algebraic equations

$$\begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for the eigenvalues and eigenvectors of $\boldsymbol{A}$. The characteristic equation is

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4) = 0;$$

therefore the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$.

Now for $\lambda_1 = -1$, Eq. (2.3) is equivalent to

$$\begin{aligned} 3w_1 + 3w_2 &= 0, \\ 2w_1 + 2w_2 &= 0. \end{aligned}$$

Thus $w_1 = -w_2$. When $w_2 = -1$, the related eigenvector is

$$\boldsymbol{v}_1 = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\lambda_2 = 4$ we have

$$\begin{aligned} -2w_2 + 3w_2 &= 0, \\ 2w_1 - 3w_2 &= 0. \end{aligned}$$

so $w_1 = 3w_2/2$, therefore with $w_2 = 2$ the corresponding eigenvector is

$$\boldsymbol{v}_2 = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Since the matrix of coefficients $\boldsymbol{A}$ is a 2×2 matrix and since we have found two linearly independent solutions of (2.3),

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \quad \text{and} \quad \boldsymbol{x}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$$

we conclude that the general solution of the system is

$$\boldsymbol{x} = C_1\boldsymbol{x}_1 + C_2\boldsymbol{x}_2 = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}.$$

Complex Eigenvalues. If some of the eigenvalues occur in complex conjugate pairs, then there are still n linearly independent solutions of the form (2.5), provided that all the eigenvalues are different. Of course, the solutions arising from complex eigenvalues are complex-valued. However, it is possible to obtain a full set of real-valued solutions.

If $\boldsymbol{A}$ is real, then the coefficients in the characteristic equation for λ are real, and any complex eigenvalues must occur in conjugate pairs. Further, the corresponding eigenvectors are also complex conjugates. To see that this is so, suppose that λ and $\boldsymbol{v}$ satisfy Eq. (2.3). On taking the complex conjugate of this equation, and noting that $\boldsymbol{A}$ and $\boldsymbol{I}$ are real-valued, we obtain

$$(\boldsymbol{A} - \lambda^* \boldsymbol{I})\boldsymbol{v}^* = 0,$$

where λ^* and $\boldsymbol{v}^*$ are the complex conjugates of λ and $\boldsymbol{v}$, respectively. In other words, λ^* is also an eigenvalue, and $\boldsymbol{v}^*$ is a corresponding eigenvector. The corresponding solutions

$$\boldsymbol{x}_1(t) = \boldsymbol{v}e^{\lambda t}, \qquad \boldsymbol{x}_2(t) = \boldsymbol{v}^* e^{\lambda^* t} \tag{2.7}$$

of the differential equations (2.1) are then complex conjugates of each other. Therefore we can find two real-valued solutions of Eq. (2.1) corresponding to the eigenvalues λ and λ^* by taking the real and imaginary parts of $\boldsymbol{x}_1(t)$ or $\boldsymbol{x}_2(t)$ given by Eq. (2.7).

Let us write $\boldsymbol{v} = \boldsymbol{p} + i\boldsymbol{q}$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are real; then we have

$$\boldsymbol{x}_1(t) = (\boldsymbol{p} + i\boldsymbol{q})e^{(\alpha + i\beta)t} = (\boldsymbol{p} + i\boldsymbol{q})e^{\alpha t}(\cos\beta t + i\sin\beta t).$$

Upon separating $\boldsymbol{x}_1(t)$ into its real and imaginary parts, we obtain

$$\boldsymbol{x}_1(t) = e^{\alpha t}(\boldsymbol{p}\cos\beta t - \boldsymbol{q}\sin\beta t) + ie^{\alpha t}(\boldsymbol{p}\sin\beta t + \boldsymbol{q}\cos\beta t).$$

If we write $\boldsymbol{x}_1(t) = \boldsymbol{u}(t) + i\boldsymbol{v}(t)$, then the vectors

$$\boldsymbol{u}(t) = e^{\alpha t}(\boldsymbol{p}\cos\beta t - \boldsymbol{q}\sin\beta t), \qquad \boldsymbol{v}(t) = e^{\alpha t}(\boldsymbol{p}\sin\beta t + \boldsymbol{q}\cos\beta t)$$

are real-valued solutions of Eq. (2.1). It is possible to show that $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent solutions.

Example 2.2. Find a fundamental set of real-valued solutions of the system

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \boldsymbol{x}.$$

To find a fundamental set of solutions we assume that $\boldsymbol{x} = \boldsymbol{v}e^{\lambda t}$ and obtain the set of linear algebraic equations

$$\begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for the eigenvalues and eigenvectors of $\boldsymbol{A}$. The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = 0;$$

therefore the eigenvalues are $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$, the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Hence a fundamental set of solutions of the origin system is

$$\boldsymbol{x}_1(t) = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(1+2i)t} \quad \text{and} \quad \boldsymbol{x}_2(t) = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(1-2i)t}.$$

To obtain a set of real-valued solutions, we must find the real and imaginary parts of either $\boldsymbol{x}_1(t)$ or $\boldsymbol{x}_2(t)$. In fact,

$$\boldsymbol{x}_1(t) = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^t(\cos 2t + i \sin 2t) = e^t \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} + ie^t \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix}.$$

Hence

$$\boldsymbol{u}(t) = e^t \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}, \qquad \boldsymbol{v}(t) = e^t \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix}$$

is a set of real-valued solutions. To verify that $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ are linearly independent, we compute their Wronskian:

$$W(\boldsymbol{u}, \boldsymbol{v}) = \begin{vmatrix} e^t \sin 2t & -e^t \cos 2t \\ e^t \cos 2t & e^t \sin 2t \end{vmatrix} = e^{2t}.$$

Since the Wronskian is never zero, it follows that $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ constitute a fundamental set of real-valued solutions of the system.

Repeated Eigenvalues. More serious difficulties can occur if an eigenvalue is repeated. Suppose that λ is an eigenvalue of multiplicity m of the matrix $\boldsymbol{A}$. In this event, there are two possibilities: either there are m linearly independent eigenvectors corresponding to the eigenvalue λ, or else there are fewer than m such eigenvectors.

In the first case, let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ be m linearly independent eigenvectors associated with the eigenvalue λ of multiplicity m. Then

$$\boldsymbol{x}_1(t) = \boldsymbol{v}_1 e^{\lambda t}, \quad \ldots, \quad \boldsymbol{x}_m(t) = \boldsymbol{v}_m e^{\lambda t}$$

are m linearly independent solutions of Eq. (2.1). Thus in this case it makes no difference that the eigenvalue λ is repeated; there is still a fundamental set of solutions of Eq. (2.1) of the form $\boldsymbol{v}e^{\lambda t}$.

In the second case, which is usually called *defective*, there will be fewer than m solutions of Eq. (2.1) of the form $\boldsymbol{v}e^{\lambda t}$ associated with this eigenvalue. Therefore, to construct the general solution of Eq. (2.1) it is necessary to find other solutions in a form involving the products of polynomials and exponential functions. So, the solution corresponding to the eigenvalue λ of multiplicity m can be written as

$$\boldsymbol{x}(t) = (\boldsymbol{w}_0 + \boldsymbol{w}_1 t + \ldots + \boldsymbol{w}_{m-k} t^{m-k}) e^{\lambda t}, \tag{2.8}$$

where the number of linearly independent eigenvectors k associated with the eigenvalue λ is given by the formula

$$k = n - \text{rank}(\boldsymbol{A} - \lambda \boldsymbol{I}). \tag{2.9}$$

To find the coefficients $\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{m-k}$, one should substitute the function (2.8) into the original system of Eq. (2.1). Equating the coefficients of the terms with the same power in the left and right sides of each equation, we obtain an algebraic system of equations for the unknown vectors $\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{m-k}$.

Example 2.3. Find the general solution of the linear system of equations

$$\begin{cases} \dot{x} = x - y, \\ \dot{y} = x + 3y. \end{cases}$$

To find the general solution of the system, we first calculate the eigenvalues of the matrix $\boldsymbol{A}$ composed of the coefficients of the equations:

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = (2-\lambda)^2 = 0.$$

Therefore, the matrix $\boldsymbol{A}$ has one eigenvalue $\lambda = 2$ of multiplicity $m = 2$. Substituting the computed eigenvalue into the expression $\boldsymbol{A} - \lambda \boldsymbol{I}$ we obtain that rank$(\boldsymbol{A} - 2\boldsymbol{I})$ is 1. Then, using the formula (2.9) we get $k = 1$. The general solution will be given by

$$\boldsymbol{x}(t) = (\boldsymbol{w}_0 + \boldsymbol{w}_1 t) e^{2t}.$$

The derivative is equal

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{w}_1 e^{2t} + 2(\boldsymbol{w}_0 + \boldsymbol{w}_1 t) e^{2t}.$$

Substitute the function $\boldsymbol{x}(t)$ and its derivative in the original system of differential equations:

$$\boldsymbol{w}_1 e^{2t} + 2(\boldsymbol{w}_0 + \boldsymbol{w}_1 t) e^{2t} = \boldsymbol{A}(\boldsymbol{w}_0 + \boldsymbol{w}_1 t) e^{2t}$$

Dividing by e^{2t} and equating the coefficients of like terms in the left and right sides, we obtain a system of algebraic equations for the coefficients $\boldsymbol{w}_0$, $\boldsymbol{w}_1$:

$$(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{w}_1 = 0 \quad \text{and} \quad (\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{w}_0 = \boldsymbol{w}_1.$$

Now the first equation is equivalent to

$$
\begin{aligned}
-u_1 - u_2 &= 0,\\
u_1 + u_2 &= 0.
\end{aligned}
$$

Thus $u_1 = -u_2$ and the vector $\boldsymbol{w}_1$ is

$$
\boldsymbol{w}_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix},
$$

where coefficient u_2 is denoted as C_1. For the second equation we have

$$
\begin{aligned}
-u_1 - u_2 &= -C_1,\\
u_1 + u_2 &= C_1.
\end{aligned}
$$

so the solution is

$$
\boldsymbol{w}_0 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix},
$$

where we denote coefficient u_2 as C_2. Thus, the general solution has the form

$$
\boldsymbol{x}(t) = \left(C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} t \right) e^{2t}.
$$

Example 2.4. Find the general solution of the system of differential equations

$$
\begin{cases}
\dot{x} = 2x + y + z,\\
\dot{y} = x + 2y + z,\\
\dot{z} = x + y + 2z.
\end{cases}
$$

To solve this system, we determine the eigenvalues of the given matrix:

$$
\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0.
$$

It can be noted that the cubic equation has a root $\lambda_1 = 1$. Factoring out the term $(\lambda - 1)$, we obtain:

$$
\lambda^3 - 6\lambda^2 + 9\lambda - 4 = \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 4\lambda - 4 = (\lambda - 1)(\lambda^2 - 5\lambda + 4) = 0
$$

The roots of the quadratic equation are: $\lambda_2 = 1$, $\lambda_3 = 4$. The initial matrix of the system is symmetric. So it will have three real eigenvectors. This means that the number of linearly independent eigenvectors associated with the eigenvalue $\lambda = 1$ equal to 2. The same result we can get using the formula (2.9):

$$k = 3 - \operatorname{rank}(\boldsymbol{A} - \lambda \boldsymbol{I}) = 3 - \operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 3 - 1 = 2.$$

Find the eigenvectors corresponding to $\lambda_{1,2} = 1$. They can be found from the equations

$$(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{w} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

We see that all three equations are identical. Leaving one equation

$$u_1 + u_2 + u_3 = 0,$$

and choosing u_2 and u_3 as independent variables, we get:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = u_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + u_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

It follows that the coordinates of the first eigenvector (with $u_2 = 1$, $u_3 = 0$) are

$$\boldsymbol{w}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Accordingly, the coordinates of the second linearly independent eigenvector (when $u_2 = 0$, $u_3 = 1$) are

$$\boldsymbol{w}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Now we define the third eigenvector $\boldsymbol{w}_3$ corresponding to the number $\lambda_3 = 4$:

$$(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{w}_3 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2u_1 + u_2 + u_3 \\ u_1 - 2u_2 + u_3 \\ u_1 + u_2 - 2u_3 \end{pmatrix} = 0.$$

Here we choose as a free variable u_3. The eigenvector $\boldsymbol{w}_3$ has the following coordinates:

$$\boldsymbol{w}_3 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The general solution of the system of differential equations is given by

$$\boldsymbol{x}(t) = C_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^t + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^t + C_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t},$$

where C_1, C_2, C_3 are arbitrary constants.

3. Nonhomogeneous Linear Systems with Constant Coefficients

So let us consider the problem of solving

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(t), \tag{3.1}$$

assuming $\boldsymbol{A}$ is some constant matrix and $\boldsymbol{f}(t)$ is a vector-valued function defined on some interval $I_{\alpha,\beta}$. We will refer to the homogeneous system

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}$$

as the corresponding or associated homogeneous system. The general solution of Eq. (3.1) has some recognizable structure.

Theorem 3.1. A general solution to a given nonhomogeneous linear system of differential equations is given by

$$\boldsymbol{x}(t) = \boldsymbol{x}_c(t) + \boldsymbol{x}_p(t),$$

where $\boldsymbol{x}_c(t)$ is a general solution to the corresponding homogeneous system, and $\boldsymbol{x}_p(t)$ is any particular solution to the nonhomogeneous equation.

This theorem assures us that we can construct a general solution for a nonhomogeneous system of differential equations from any single particular solution $\boldsymbol{x}_p(t)$, provided we know a general solution $\boldsymbol{x}_c(t)$ for the corresponding homogeneous system.

Another important property of the systems (3.1) is the principle of superposition, which is formulated as follows:

Theorem 3.2. If $\boldsymbol{x}_1(t)$ is a solution of the system with the inhomogeneous part $\boldsymbol{f}_1(t)$, and $\boldsymbol{x}_2(t)$ is a solution of the same system with the inhomogeneous part $\boldsymbol{f}_2(t)$, then the vector function

$$\boldsymbol{x}(t) = \boldsymbol{x}_1(t) + \boldsymbol{x}_2(t)$$

is a solution of the system with the inhomogeneous part

$$\boldsymbol{f}(t) = \boldsymbol{f}_1(t) + \boldsymbol{f}_2(t).$$

The most common methods of solution of the nonhomogeneous systems are the method of undetermined coefficients (in the case where the function $\boldsymbol{f}(t)$ is a vector quasi-polynomial), and the method of variation of parameters. Consider these methods in more detail.

Undetermined Coefficients. The method of undetermined coefficients is well suited for solving systems of equations, the nonhomogeneous part of which is a quasi-polynomial.

A real vector quasi-polynomial is a vector function of the form

$$\boldsymbol{f}(t) = e^{\alpha t}(\boldsymbol{P}_m(t)\cos\beta t + \boldsymbol{Q}_m(t)\sin\beta t), \tag{3.2}$$

where α, β are given real numbers, and $\boldsymbol{P}_m(t)$, $\boldsymbol{Q}_m(t)$ are vector polynomials of degree m. For example, a vector polynomial $P_m(t)$ is written as

$$\boldsymbol{P}_m(t) = \boldsymbol{p}_0 + \boldsymbol{p}_1 t + \ldots + \boldsymbol{p}_m t^m, \tag{3.3}$$

where $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_m$ are n-dimensional vectors (n is the number of equations in the system).

In the case when the nonhomogeneous part $\boldsymbol{f}(t)$ is a vector quasi-polynomial, a particular solution is also given by a vector quasi-polynomial, similar in structure to $\boldsymbol{f}(t)$.

For example, if the nonhomogeneous function is

$$\boldsymbol{f}(t) = e^{\alpha t}\boldsymbol{P}_m(t)$$

a particular solution should be sought in the form

$$\boldsymbol{x}_p(t) = e^{\alpha t}\boldsymbol{P}_{m+r}(t)$$

where $r = 0$ in the non-resonance case, i. e. when the index α in the exponential function does not coincide with an eigenvalue λ_k. If the index α coincides with an eigenvalue λ_k, i. e. in the so-called resonance case, the value of r is set equal to the length of the Jordan chain for the eigenvalue λ_k. In practice, r can be taken as the algebraic multiplicity of λ_k.

Similar rules for determining the degree of the polynomials are used for quasi-polynomials of kind

$$\boldsymbol{f}(t) = e^{\alpha t}(\boldsymbol{P}_m(t)\cos\beta t + \boldsymbol{Q}_n(t)\sin\beta t)$$

Here the resonance case occurs when the number $\alpha + \beta i$ coincides with a complex eigenvalue λ_k of the matrix $\boldsymbol{A}$.

After the structure of a particular solution $\boldsymbol{x}_p(t)$ is chosen, the unknown vector coefficients $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_{m+r}$ are found by substituting the expression for $\boldsymbol{x}_p(t)$ in the original system and equating the coefficients of the terms with equal powers of t on the left and right side of each equation.

Example 3.1. Solve the system of equations by the method of undetermined coefficients:

$$\begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 3y + te^t. \end{cases}$$

We rewrite this system in matrix form:

$$\dot{\boldsymbol{x}}(t) = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 0 \\ te^t \end{pmatrix}$$

The general solution of the homogeneous system is given by

$$\boldsymbol{x}_c(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

Now we find a particular solution $\boldsymbol{x}_p(t)$. The inhomogeneous term has the form of a quasi-polynomial $\boldsymbol{P}_1(t)e^t$. The degree of the exponential function is $\alpha = 1$. Because it does not coincide with any of the eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$, then we find a particular solution in a form similar to $\boldsymbol{f}(t)$, i. e. assume that

$$\boldsymbol{x}_p(t) = (\boldsymbol{p}_0 + \boldsymbol{p}_1 t)e^t.$$

We find the unknown vectors $\boldsymbol{p}_0$, $\boldsymbol{p}_1$ using the method of undetermined coefficients. Let

$$\boldsymbol{p}_0 = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{p}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}.$$

Substitute $\boldsymbol{x}_p(t)$ into the original nonhomogeneous equation:

$$\begin{aligned} (a_1 + a_0 + a_1 t)e^t &= (2a_0 + 2a_1 t)e^t + (b_0 + b_1 t)e^t, \\ (b_1 + b_0 + b_1 t)e^t &= (3b_0 + 3b_1 t)e^t + te^t \end{aligned}$$

Divide both sides of each equation by e^t:

$$\begin{aligned} a_1 + a_0 + a_1 t &= 2a_0 + 2a_1 t + b_0 + b_1 t, \\ b_1 + b_0 + b_1 t &= 3b_0 + 3b_1 t + t \end{aligned}$$

By equating the coefficients of like terms, we obtain the following system of equations:

$$\begin{aligned} a_1 + a_0 &= 2a_0 + b_0, \\ a_1 &= 2a_1 + b_1, \\ b_1 + b_0 &= 3b_0, \\ b_1 &= 3b_1 + 1 \end{aligned}$$

Solve the system and find the unknown coefficients a_0, a_1, b_0, b_1:

$$a_0 = \frac{3}{4}, \qquad a_1 = \frac{1}{2}, \qquad b_0 = -\frac{1}{4}, \qquad b_1 = -\frac{1}{2}.$$

Thus, a particular solution $\boldsymbol{x}_p(t)$ can be written as:

$$\boldsymbol{x}_p(t) = \frac{1}{4} e^t \begin{pmatrix} 3 + 2t \\ -1 - 2t \end{pmatrix}.$$

Then the general solution of the original nonhomogeneous system is given by the following formula:

$$\boldsymbol{x}(t) = \boldsymbol{x}_c(t) + \boldsymbol{x}_p(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \frac{1}{4} e^t \begin{pmatrix} 3+2t \\ -1-2t \end{pmatrix}$$

Variation of Parameters. The method of variation of parameters or Lagrange method was introduced by the Swiss-born mathematician Leonhard Euler and completed by the Italian-French mathematician Joseph-Louis Lagrange. It is the common method of solution in the case of an arbitrary right-hand side $\boldsymbol{f}(t)$.

Suppose that the general solution of the associated homogeneous system is found and represented as

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}, \tag{3.4}$$

where $\boldsymbol{\Phi}(t)$ is a fundamental matrix whose columns are formed by linearly independent solutions of the homogeneous system, and $\boldsymbol{c}$ is a constant vector. It is natural to seek a solution of the nonhomogeneous system (3.1) by replacing the constant vector $\boldsymbol{c}$ by a vector function $\boldsymbol{u}(t)$. Thus, we assume that

$$\boldsymbol{x} = \boldsymbol{\Phi}(t)\boldsymbol{u}(t), \tag{3.5}$$

where $\boldsymbol{u}(t)$ is a vector to be found. Upon requiring that Eq. (3.1) be satisfied, we obtain

$$\boldsymbol{\Phi}'(t)\boldsymbol{u}(t) + \boldsymbol{\Phi}(t)\boldsymbol{u}'(t) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t)\boldsymbol{u}(t) + \boldsymbol{f}(t). \tag{3.6}$$

Since $\boldsymbol{\Phi}(t)$ is a fundamental matrix, $\boldsymbol{\Phi}'(t) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t)$; hence Eq. (3.6) reduces to

$$\boldsymbol{\Phi}(t)\boldsymbol{u}'(t) = \boldsymbol{f}(t).$$

There exists the inverse matrix $\boldsymbol{\Phi}^{-1}(t)$ because the Wronskian of the system is not equal to zero. Multiplying the last equation on the left by $\boldsymbol{\Phi}^{-1}(t)$, we obtain:

$$\boldsymbol{u}'(t) = \boldsymbol{\Phi}^{-1}(t)\boldsymbol{f}(t). \tag{3.7}$$

Vector function $\boldsymbol{u}(t)$ can be determined only up to an arbitrary constant vector $\boldsymbol{c}$:

$$\boldsymbol{u} = \int \boldsymbol{\Phi}^{-1}(s)\boldsymbol{f}(s)ds + \boldsymbol{c}. \tag{3.8}$$

Finally, substituting for $\boldsymbol{u}$ in Eq. (3.5) gives the general solution $\boldsymbol{x}$ of the nonhomogeneous system (3.1):

$$\boldsymbol{x} = \boldsymbol{\Phi}(t)\boldsymbol{u}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c} + \boldsymbol{\Phi}(t)\int \boldsymbol{\Phi}^{-1}(s)\boldsymbol{f}(s)ds. \tag{3.9}$$

Thus, the solution of the nonhomogeneous equation can be expressed in quadratures for any inhomogeneous term $\boldsymbol{f}(t)$. In many problems, the corresponding integrals can be calculated analytically. This allows us to express the solution of the nonhomogeneous system explicitly.

Example 3.2. Find the general solution to the following system

$$\begin{cases} \dot{x} = 2x + 3y + \cos t, \\ \dot{y} = 2x + y. \end{cases}$$

The general solution of the associated homogeneous system was found in the example 2.1, the fundamental matrix $\mathbf{\Phi}(t)$ and its inverse $\mathbf{\Phi}^{-1}(t)$ are,

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{pmatrix}, \qquad \mathbf{\Phi}^{-1}(t) = \frac{1}{5}\begin{pmatrix} 2e^{t} & -3e^{t} \\ e^{-4t} & e^{-4t} \end{pmatrix}.$$

Now we calculate the multiplication in the integral

$$\mathbf{\Phi}^{-1}(t)\boldsymbol{f}(t) = \frac{1}{5}\begin{pmatrix} 2e^{t} & -3e^{t} \\ e^{-4t} & e^{-4t} \end{pmatrix}\begin{pmatrix} \cos t \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2e^{t}\cos t \\ e^{-4t}\cos t \end{pmatrix}.$$

Then we have to integrate it. Recall that to integrate a matrix or vector we just integrate the individual entries.

$$\int \frac{1}{5}\begin{pmatrix} 2e^{t}\cos t \\ e^{-4t}\cos t \end{pmatrix} dt = \frac{1}{5}\begin{pmatrix} 2\displaystyle\int e^{t}\cos t dt \\ \displaystyle\int e^{-4t}\cos t dt \end{pmatrix} = \frac{1}{85}\begin{pmatrix} 17e^{t}(\cos t + \sin t) \\ e^{-4t}(\sin t - 4\cos t) \end{pmatrix}.$$

The general solution is then,

$$\boldsymbol{x} = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} + \frac{1}{85}\begin{pmatrix} 17e^{t}(\cos t + \sin t) \\ e^{-4t}(\sin t - 4\cos t) \end{pmatrix}.$$

4. *Exercises*

1. Solve the given systems of the linear differential equations:

1) $\begin{cases} \dot{x} = 2x + y \\ \dot{y} = 3x + 4y \end{cases}$

2) $\begin{cases} \dot{x} = x - y \\ \dot{y} = x - 4y \end{cases}$

3) $\begin{cases} \dot{x} = x + y \\ \dot{y} = 3x - 2y \end{cases}$

4) $\begin{cases} \dot{x} = 3x - y \\ \dot{y} = 4x - y \end{cases}$

5) $\begin{cases} \dot{x} = 2x + y \\ \dot{y} = 4x - y \end{cases}$

6) $\begin{cases} \dot{x} = 2y - 3x \\ \dot{y} = x - 2y \end{cases}$

7) $\begin{cases} \dot{x} = -x + 8y \\ \dot{y} = x + y \end{cases}$

8) $\begin{cases} \dot{x} = 5x + 3y \\ \dot{y} = -3x - y \end{cases}$

9) $\begin{cases} \dot{x} = x + z - y \\ \dot{y} = x + y - z \\ \dot{z} = 2x - y \end{cases}$

10) $\begin{cases} \dot{x} = x - 2y - z \\ \dot{y} = y - x + z \\ \dot{z} = x - z \end{cases}$

11) $\begin{cases} \dot{x} = 2x - y + z \\ \dot{y} = x + 2y - z \\ \dot{z} = x - y + 2z \end{cases}$

12) $\begin{cases} \dot{x} = 3x - y + z \\ \dot{y} = x + y + z \\ \dot{z} = 4x - y + 4z \end{cases}$

13) $\begin{cases} \dot{x} = x - y - z \\ \dot{y} = x + y \\ \dot{z} = 3x + z \end{cases}$

14) $\begin{cases} \dot{x} = 4x - y - z \\ \dot{y} = x + 2y - z \\ \dot{z} = x - y + 2z \end{cases}$

15) $\begin{cases} \dot{x} = 2x - y - z \\ \dot{y} = 3x - 2y - 3z \\ \dot{z} = 2z - x + z \end{cases}$

16) $\begin{cases} \dot{x} = 3x - 2y - z \\ \dot{y} = 3x - 4y - 3z \\ \dot{z} = 2x - 4y \end{cases}$

17) $\begin{cases} \dot{x} = 2x + y \\ \dot{y} = 2y + 4z \\ \dot{z} = x - z \end{cases}$

18) $\begin{cases} \dot{x} = 2x - y - z \\ \dot{y} = 2x - y - 2z \\ \dot{z} = 2z - x + y \end{cases}$

19) $\begin{cases} \dot{x} = 4x - y \\ \dot{y} = 3x + y - z \\ \dot{z} = x + y \end{cases}$

2. Solve the systems of equations:

1) $\begin{cases} \ddot{x} = 2x - 3y \\ \ddot{y} = x - 2y \end{cases}$

2) $\begin{cases} \ddot{x} = 3x + 4y \\ \ddot{y} = -x - y \end{cases}$

3) $\begin{cases} \ddot{x} = 2y \\ \ddot{y} = -2x \end{cases}$

4) $\begin{cases} 2\dot{x} - 5\dot{y} = 4y - x \\ 3\dot{x} - 4\dot{y} = 2x - y \end{cases}$

5) $\begin{cases} \ddot{x} + \dot{x} + \dot{y} = 2y \\ \ddot{x} - \dot{y} + x = 0 \end{cases}$

6) $\begin{cases} \ddot{x} - 2\ddot{y} = 3y - x - \dot{y} \\ 4\ddot{y} - 2\ddot{x} = \dot{x} + 2x - 5y \end{cases}$

7) $\begin{cases} \ddot{x} = 3x - y - z \\ \ddot{y} = -x + 3y - z \\ \ddot{z} = -z - y + 3z \end{cases}$

3. Use the method of undetermined coefficients or variation of parameters to solve the given systems:

1) $\begin{cases} \dot{x} = x + 3y - 2t^2 \\ \dot{y} = 3x + y + t + 5 \end{cases}$

2) $\begin{cases} \dot{x} = x - 4y + 4t + 9e^{6t} \\ \dot{y} = 4x + y - t + e^{6t} \end{cases}$

3) $\begin{cases} \dot{x} = x + 2y \\ \dot{y} = x - 5\sin t \end{cases}$

4) $\begin{cases} \dot{x} = 2x - 4y \\ \dot{y} = x - 3y + 3e^t \end{cases}$

5) $\begin{cases} \dot{x} = x + 2y + 16te^t \\ \dot{y} = 2x - 2y \end{cases}$

6) $\begin{cases} \dot{x} = 2x - y \\ \dot{y} = y - 2x + 18t \end{cases}$

7) $\begin{cases} \dot{x} = 2x + 4y - 8 \\ \dot{y} = 3x + 6y \end{cases}$

8) $\begin{cases} \dot{x} = 2x - 3y \\ \dot{y} = x - 2y + 2\sin t \end{cases}$

9) $\begin{cases} \dot{x} = x - y + 2\sin t \\ \dot{y} = 2x - y \end{cases}$

10) $\begin{cases} \dot{x} = 2x - y \\ \dot{y} = x + 2e^t \end{cases}$

11) $\begin{cases} \dot{x} = 4x - 3y + \sin t \\ \dot{y} = 2x - y - 2\cos t \end{cases}$

12) $\begin{cases} \dot{x} = 2x + y + 2e^t \\ \dot{y} = x + 2y - 3e^{4t} \end{cases}$

13) $\begin{cases} \dot{x} = x - y + 8t \\ \dot{y} = 5x - y + \sin 2t \end{cases}$

14) $\begin{cases} \dot{x} = 2x - y \\ \dot{y} = 2y - x - 5e^t \sin t \end{cases}$

4. Use the method of variation of parameters to solve the given systems:

1) $\begin{cases} \dot{x} = y + \tan^2 t - 1 \\ \dot{y} = -x + \tan t \end{cases}$

2) $\begin{cases} \dot{x} = 2y - x \\ \dot{y} = 4y - 3x + \dfrac{e^{3t}}{e^{2t} + 1} \end{cases}$

3) $\begin{cases} \dot{x} = x - y + \dfrac{1}{\cos t} \\ \dot{y} = 2x - y \end{cases}$

4) $\begin{cases} \dot{x} = -4x - 2y + \dfrac{2}{e^t - 1} \\ \dot{y} = 6x + 3y + \dfrac{3}{e^t - 1} \end{cases}$

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