МИНИСТЕРСТВО ОБРАЗОВАНИЯ И НАУКИ РФ

**Нижегородский государственный университет им. Н.И. Лобачевского**

# МЕТОДЫ ИНТЕГРИРОВАНИЯ В ПРИМЕРАХ И ЗАДАЧАХ

*Учебно-методическое пособие*

Рекомендовано методической комиссией факультета ВМК для иностранных студентов, обучающихся в ННГУ по направлению подготовки 010300 "Фундаментальная информатика и информационные технологии"(бакалавриат).

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Пособие представляет собой сборник материалов по элементарным методам вычисления неопределенных и определенных интегралов. В пособие включены необходимые определения и факты, детальный разбор примеров и задачи для самостоятельной проработки. Предназначено для учащихся факультета иностранных студентов по направлению подготовки 010300 "Фундаментальная информатика и информационные технологии".

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# METHODS OF INTEGRATION WITH EXAMPLES AND TASKS

*Study book*

Recommended by the Methodical Commission of the Faculty of Computer Science for international students, studying at the B.Sc. programme 010300 "Fundamental Informatics and Information Technologies".

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# Contents

## Introductory Remarks

We discuss a range of techniques for computing integrals (indefinite and definite) which usually cause students difficulties. This textbook does not cover all material on integrals be to considered during classes. However, it presents a detailed introduction to the subject.

CHAPTER 1

# Indefinite Integrals

## 1. Antiderivative and Indefinite Integrals

We wish to perform the opposite process to differentiation. It's naturally to call it "antidifferentiation".

Definition 1.1. *A function $F$ is an antiderivative of a function $f$ if $F' = f$.*

**Example:** Find antiderivative of function $f = 3x^2$.

Solution: We are going to find a function $F$ whose derivative equals to $f = 3x^2$, so we have:

$F' = f = 3x^2$.

Think: "What would I have to differentiate to get $3x^2$?"

$F = x^3$ is ONE antiderivative of $3x^2$.

In fact, there are infinitely many other antiderivatives which would also work, for example:

$F = x^3 + 4$;

$F = x^3 - 10$;

$F = x^3 + 27.3$.

This fact is reflection of the following theorem.

Theorem 1.1. *Assume that $f$ is continuous on an interval $(a, b)$. If $F_1$ and $F_2$ are both antiderivatives of the function $f$ on the interval $(a, b)$, then there is a constant $C$ such that $F_1 = F_2 + C$.*

**Proof:** Consider the function $F = F_1 - F_2$. By the properties of $F_1$ and $F_2$, $F' = F_1' - F_2' = f - f = 0$. Then, $F$ is a constant function. $\diamond$

DEFINITION 1.2. *The set of all antiderivatives to a function $f(x)$ is called indefinite integral of $f(x)$ and denoted by $\int f(x)dx$. If $F(x)$ is any antiderivative of $f(x)$, then*

$$\int f(x)dx = F(x) + C,$$

*where $C$ is any constant.*

In this definition the $\int$ is called the integral symbol, $f(x)$ is called the integrand, $x$ is called the integration variable and the $C$ is called the constant of integration.

Note that often we will just say integral instead of indefinite integral (or definite integral for that matter when we get to those). It will be clear from the contest of the problem that we are talking about an indefinite integral (or definite integral).

The process of finding the indefinite integral is called integration or integrating $f(x)$. If we need to be specific about the integration variable we will say that we are integrating $f(x)$ with respect to $x$.

**Example:** Due to previous example, one can write

$$\int 3x^2dx = x^3 + C.$$

Warning! One of the more common mistakes that students make with integrals (both indefinite and definite) is to drop the $dx$ at the end of the integral. This is required! Think of the integral sign and the $dx$ as a set of parentheses. You already know and are probably quite comfortable with the idea that every time you open a parenthesis you must close it. With integrals, think of the integral sign as an "open parenthesis" and the $dx$ as a "close parenthesis".

Another use of the differential at the end of integral is to tell us what variable we are integrating with respect to. At this stage that may seem unimportant since most of the integrals that we're going to be working with here will only involve a single variable. However, if you are on a degree track that will take you into multi-variable calculus this will be very important at that stage since there will be more than one variable in the problem.

To see why this is important take a look at the following two integrals.

$$\int 2x dx, \quad \int 2t dx.$$

The first integral is simple enough:

$$\int 2x dx = x^2 + C.$$

The second integral is also fairly simple, but we need to be careful. The $dx$ tells us that we are integrating $x$'s. That means that we only integrate $x$'s that are in the integrand and all other variables in the integrand are considered to be constants. The second integral is then,

$$\int 2t dx = 2tx + C.$$

So, it may seem silly to always put in the $dx$, but it is a vital bit of notation that can cause us to get the incorrect answer if we neglect to put it in.

Now, there are some important properties of integrals that we should take a look at.

**Linearity of the Indefinite Integral**

$\int kf(x)dx = k\int f(x)dx$ where $k$ is any number.

$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

So, we can factor multiplicative constants out of indefinite integrals and the integral of a sum of functions is the sum of the individual integrals. This rule can be extended to as many functions as we need.

**Exercises:** Use table of derivatives to find integrals below.

**1.1.1.** $\int (x^2-5)dx$;
**1.1.2.** $\int 4dx$;
**1.1.3.** $\int \frac{1}{\sqrt{x}}dx$.

## 2. Computing Indefinite Integrals

The simplest indefinite integrals may be evaluated using linearity of integrals (what allows to cut integral of sum of functions into several individual integrals and to factor multiplicative constants out of indefinite integrals), and table of derivatives, which allows to "differentiate backward" and find antiderivative of given function. To simplify this process we present here the table of basic indefinite integrals and then show how to use it.

| |
|---|
| $\int x^n dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1)$ |
| $\int \frac{dx}{x} = ln\|x\| + C$ |
| $\int \sin x dx = -\cos x + C$ |
| $\int \cos x dx = \sin x + C$ |
| $\int \frac{1}{\cos^2 x} dx = \tan x + C$ |
| $\int \frac{1}{\sin^2 x} dx = \cot x + C$ |
| $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$ |
| $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + C\ (\|x\| < a)$ |
| $\int e^x dx = e^x + C$ |
| $\int a^x dx = \frac{a^x}{\ln\|a\|} + C$ |
| $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln\|\frac{a+x}{a-x}\| + C\ (\|x\| < a)$ |
| $\int \frac{1}{\sqrt{a^2+x^2}} dx = \ln\|x + \sqrt{x^2+a^2}\| + C$ |
| $\int \frac{1}{\sqrt{a^2-x^2}} dx = \ln\|x + \sqrt{x^2-a^2}\| + C$ |

Let's look at the first integral of the table:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1.$$

The general rule when integrating a power of $x$ we add one onto the exponent and then divide by the new exponent. It is clear that we will need to avoid $n=-1$ in this formula: if we allow $n=-1$ in this formula we will end up with division by zero.

One of the application of this formula is the following integral (where $k$ is any number):

$$\int k dx = k \int dx = k \int x^0 dx = k\frac{x^{0+1}}{0+1} + C = kx + C.$$

**Example:** Find $\int (4x - 2\sin x + \frac{3}{\sqrt{1-x^2}})dx$

Solution: First, we use linearity of indefinite integral to cut it into three integrals, then apply the table of integrals:

$\int (4x - 2\sin x + \frac{3}{\sqrt{1-x^2}})dx = 4\int xdx - 2\int \sin xdx + 3\int \frac{dx}{\sqrt{1-x^2}} = 4\frac{x^2}{2} - (-\cos x) + 3\arcsin x + C = 2x^2 + 2\cos x + 3\arcsin x + C.$

**Example:** Find $\int (\sqrt{x} + \frac{1}{\sqrt{x}})dx.$

Solution: We use property of exponential functions to transform the squar roots under the integrals, then apply table of integrals:

$$\sqrt{x} = x^{\frac{1}{2}};\ \frac{1}{\sqrt{x}} = (\sqrt{x})^{-1} = (x^{\frac{1}{2}})^{-1} = x^{-\frac{1}{2}};$$

$$\int (\sqrt{x} + \frac{1}{\sqrt{x}})dx = \int (x^{\frac{1}{2}} + x^{-\frac{1}{2}})dx = \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx =$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C =$$

$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C = \frac{2}{3}x\sqrt{x} + 2\sqrt{x} + C.$$

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**<u>Exercises:</u>** Find integrals below.

**1.2.1.** $\int (3e^x - 5\sin x + 2\sqrt{x})dx;$

**1.2.2.** $\int (\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x^2}})dx;$

**1.2.3.** $\int \frac{dx}{8-2x^2};$

**1.2.4.** $\int \frac{1}{\sqrt[3]{x^2}}dx;$

## 3. Substitution Method of Integration

The method of substitution is a method for algebraically simplifying the form of a function so that its antiderivative can be easily recognized.

**Example:** Integrate $\int (2x-3)^{100}dx$.

Solution: Let's put $u = 2x - 3$, then $du = (2x-3)'dx = 2dx$; so $dx = \frac{1}{2}du$.

Replacing $2x - 3$ with $u$ and $dx$ with $\frac{1}{2}du$ in the integral, we get: $\int (2x-3)^{100}dx = \int u^{100}\frac{1}{2}du$.

Now we only need to apply table of integrals to evaluate last integral and to come back to $x$:

$\int (2x-3)^{100}dx = \int u^{100}\frac{1}{2}du = \frac{1}{2}\frac{u^{100+1}}{100+1} + C = \frac{u^{101}}{202} + C = \frac{(2x-3)^{101}}{202} + C$.

◇

Let's now review the five steps for integration by substitution.

Step 1: Choose a new variable $u$
Step 2: Determine the value $du$
Step 3: Make the substitution
Step 4: Integrate resulting integral
Step 5: Return to the initial variable $x$

The key to choose proper substitution is the following Chain Rule of differentiation:

$$(f(g(x)))' = f'(g(x))g'(x)$$

The Chain Rule implies

$$\int f'(g(x))g'(x)dx = \int (f(g(x)))'dx = f(g(x)) + C.$$

Usually the above rule is written slightly differently as follows:

$$\boxed{\begin{array}{l} \int f(g(x))g'(x)dx = \int f(u)du, \ where \\ \\ u = g(x), \ du = g'(x)dx. \end{array}}$$

**Example:** Integrate $\int (2x+2)e^{x^2+2x+3}dx$.

Solution: Note that the derivative of $e^{x^2+2x+3}$ can be computed using the chain rule and is $(e^{x^2+2x+3})' = e^{x^2+2x+3}(2x+2)$.

Thus, it follows easily that $\int (2x+2)e^{x^2+2x+3}dx = e^{x^2+2x+3} + C$.

This is an illustration of the Chain Rule "backwards". Now the method of substitution will be illustrated on this same example.

Begin with $\int (2x+2)e^{x^2+2x+3}dx$ and let

$$u = x^2 + 2x + 3.$$

Then the derivative of $u$ is

$$u' = \frac{du}{dx} = 2x + 2.$$

Now "pretend" that the differentiation notation $\frac{du}{dx}$ is an arithmetic fraction, and multiply both sides of the previous equation by $dx$:

$$du = \frac{du}{dx}dx = (2x+2)dx.$$

Make substitutions into the original problem, removing all forms of $x$, resulting in

$$\int (2x+2)e^{x^2+2x+3}dx = \int e^u du = e^u + C = e^{x^2+2x+3} + C.$$

Of course, it is the same answer that we got before, using the chain rule "backwards". In essence, the method of substitution is a way to recognize the antiderivative of a chain rule derivative. $\diamond$

**Example:** To compute $\int 2xe^{x^2}dx$ choose substitution $u = x^2$. Then $du = 2xdx$, $\int 2xe^{x^2}dx = \int e^u du = e^u + C = e^{x^2} + C$.

The function to be integrated suggested the substitution $u = x^2$ because the derivative of $x^2$, $2x$, is a factor of the function to be integrated. $\diamond$

<u>**Exercises:**</u> Integrate

**1.3.1.** $\int 3(8y-1)e^{4y^2-y}dy$;

**1.3.2.** $\int x^2(3-10x^3)^4dx$;

**1.3.3.** $\int \frac{x}{\sqrt{1-4x^2}}dx$;

**1.3.4.** $\int (1-\frac{1}{x})\cos(x-\ln x)dx$.

**Example:** Integrate

(1) $\int \frac{3}{5y+4}dy$;

(2) $\int \frac{3y}{5y^2+4}dy$;

(3) $\int \frac{3y}{(5y^2+4)^2}dy$;

(4) $\int \frac{3}{5y^2+4}dy$.

Solution:

(1) $\int \frac{3}{5y+4}dy$.

Let's notice that if we take the denominator and differentiate it we get just a constant and the only thing that we have in the numerator is also a constant. This is a pretty good indication that we can use the denominator for our substitution so,

$$u = 5y+4,\ du = 5dy,\ dy = \frac{1}{5}du.$$

The integral is now

$$\int \frac{3}{5y+4}dy = 3\int \frac{1}{5y+4}dy = 3\int \frac{1}{u}\frac{1}{5}du =$$
$$= \frac{3}{5}\int \frac{du}{u} = \frac{3}{5}\ln|u| + C = \frac{3}{5}\ln|5y+4| + C.$$

Remember that we can just factor the 3 in the numerator out of the integral and that makes the integral a little clearer in this case.

(2) $\int \frac{3y}{5y^2+4}dy$.

The integral is very similar to the previous one with a couple of minor differences but notice that again if we differentiate the denominator we get something that is different from the numerator by only a multiplicative constant. Therefore we'll again take the denominator as our substitution.

$$u = 5y^2 + 4, \ du = 10ydy, \ ydy = \frac{1}{10}du.$$

The integral is

$$\int \frac{3y}{5y^2+4}dy = \frac{3}{10}\int \frac{du}{u} = \frac{3}{10}\ln|u| + C =$$
$$= \frac{3}{10}\ln|5y^2+4| + C.$$

(3) $\int \frac{3y}{(5y^2+4)^2}dy$.

Now, this one is almost identical to the previous part except we added a power onto the denominator. Notice however that if we ignore the power and differentiate what's left we get the same thing as the previous example so we'll use the same substitution here.

$$u = 5y^2 + 4, \ du = 10ydy, \ ydy = \frac{1}{10}du.$$

$$\int \frac{3y}{(5y^2+4)^2}dy = \frac{3}{10}\int \frac{du}{u^2} = -\frac{3}{10}u^{-1} + C =$$
$$= -\frac{3}{10}(5y^2+4)^{-1} + C = -\frac{3}{10(5y^2+4)} + C.$$

(4) $\int \frac{3}{5y^2+4}dy$.

Now, this part is completely different from the first three and yet seems similar to them as well. In this case if we differentiate the denominator we get a $y$ that is not in the numerator and so we can't use the denominator as our substitution. In fact, because we have $y^2$ in the denominator and no $y$ in the numerator is an indication of how to work this problem. This integral is going to be an inverse tangent when we are done. To key to seeing this is to recall the following formula:

$$\int \frac{1}{u^2+1}du = \arctan u + C$$

We clearly don't have exactly this but we do have something that is similar. The denominator has a squared term plus a constant and the numerator is just a constant. So, with a little work and the proper substitution we should be able to get our integral into a form that will allow us to use this formula. There is one part of this formula that is really important and that is the "+1" in the denominator. That must be there and we've got a "+4" but it is easy enough to take care of that. We'll just factor a 4 out of the denominator and at the same time we'll factor the 3 in the numerator out of the integral as well. Doing this gives

$$\int \frac{3}{5y^2+4}dy = \int \frac{3}{4(\frac{5}{4}y^2+1)}dy = \frac{3}{4}\int \frac{1}{\frac{5y^2}{4}+1}dy =$$

$$= \frac{3}{4}\int \frac{1}{(\frac{\sqrt{5}y}{2})^2+1}dy$$

Notice that in the last step we rewrote things a little in the denominator. This will help us to see what the substitution needs to be. In order to get this integral into the formula above we need to end up with a $u^2$ in the denominator. Our substitution will then need to be something that upon squaring gives us $\frac{\sqrt{5}y}{2}$. With the rewrite we can see what that we'll need to use the following substitution

$$u = \frac{\sqrt{5}y}{2},\ du = \frac{\sqrt{5}}{2}dy,\ dy = \frac{2}{\sqrt{5}}du.$$

Upon plugging our substitution in we get

$$\frac{3}{4}\int \frac{1}{(\frac{\sqrt{5}y}{2})^2+1}dy = \frac{3}{4}\left(\frac{2}{\sqrt{5}}\right)\int \frac{1}{u^2+1}du =$$

$$= \frac{3}{2\sqrt{5}}\int \frac{1}{u^2+1}du = \frac{3}{2\sqrt{5}}\arctan u + C = \frac{3}{2\sqrt{5}}\arctan\frac{\sqrt{5}y}{2} + C.$$

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**Exercises:** Integrate
**1.3.5.** $\int \frac{2t^3+1}{(t^4+2t)^3}dt$;
**1.3.6.** $\int \frac{2t^3+1}{t^4+2t}dt$;
**1.3.7.** $\int \frac{x}{\sqrt{1-4x^2}}dt$;
**1.3.8.** $\int \frac{1}{\sqrt{1-4x^2}}dt$.

**Example: Trig substitution.** Compute $\int \sqrt{1-x^2}dx$.
Use the substitution $x = \sin t$ because we can then get rid of the square root since $\sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \cos t$.
We have
$dx = \cos t dt$ and $\int \sqrt{1-x^2}dx = \int \cos^2 t dt$.
To compute the last integral we use the formula $\cos^2 t = \frac{1+\cos(2t)}{2}$.
One gets

$$\int \cos^2 t dt = \frac{1}{2}\int (1+\cos(2t))dx = \frac{1}{2}(t+\frac{1}{2}\sin 2t) + C =$$

$$= \frac{1}{2}(t+\sin t\cos t) + C = \frac{1}{2}(\arcsin x + x\sqrt{1-x^2}) + C.$$

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**Exercises:** Integrate
**1.3.9.** $\int \frac{dx}{x^2\sqrt{x^2-9}}$;
**1.3.10.** $\int \frac{\sqrt{1-x^2}}{x^2}dx$;
**1.3.11.** $\int x^2\sqrt{4-x^2}dx$.

## 4. Integration by Parts

Method of integrations by parts is based on the Product Rule formula:

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x).$$

Now, let's integrate both sides of this:

$$\int (u(x)v(x))' dx = \int u'(x)v(x)dx + \int u(x)v'(x)dx.$$

The left side is easy enough to integrate and we'll split up the right side of the integral.

$$u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)dx.$$

Note that technically we should have had a constant of integration show up on the left side after doing the integration. We can drop it at this point since other constants of integration will be showing up down the road and they would just end up absorbing this one.

Finally, rewrite the formula as follows and we arrive at the integration by parts formula:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

Denote than $v'(x)dx = dv$, $u'(x)dx = du$ to get the simplest form of the formula:

$$\boxed{\int udv = uv - \int vdu}$$

To use the method we will need to identify $u$ and $dv$, compute $du$ and $v$ and then use the formula. Note as well that computing $v$ is very easy. All we need to do is integrate $dv$: $v = \int dv$.

**Example:** Evaluate the integral $\int xe^{6x}dx$.

Solution. So, on some level, the problem here is the $x$ that is in front of the exponential. If that wasn't there we could do the integral. Notice as well that in doing integration by parts anything that we choose for $u$ will be differentiated. So, it seems that choosing $u = x$ will be a good choice since upon differentiating the $x$ will drop out.

Now that we've chosen $u$ we know that $dv$ will be everything else that remains. So, here are the choices for $u$ and $dv$ as well as $du$ and $v$.

$$u = x,\ dv = e^{6x}dx;$$

$$du = dx,\ v = \int e^{6x}dx = \frac{1}{6}e^{6x}.$$

The integral is then

$$\int xe^{6x}dx = \frac{x}{6}e^{6x} - \frac{1}{6}\int e^{6x}dx = \frac{x}{6}e^{6x} - \frac{1}{36}e^{6x} + C.$$

Once we have done the last integral in the problem we will add in the constant of integration to get our final answer. ◇

**Exercises:** Integrate

**1.4.1.** $\int x sin 2x dx$;
**1.4.2.** $\int x3^x dx$;
**1.4.3.** $\int xe^{-x}dx$;
**1.4.4.** $\int x \arctan x dx$;
**1.4.5.** $\int \arccos x dx$;
**1.4.6.** $\int x \cos^2 x dx$;
**1.4.7.** $\int \frac{\ln x}{x^3}dx$;
**1.4.8.** $\int \ln(x^2+1)dx$;
**1.4.9.** $\int x^2e^{-x}dx$;
**1.4.10.** $\int e^x \sin x dx$.

CHAPTER 2

# Definite Integrals

## 1. Area of Domain and Definite Integral

**Example:** Determine the area of the domain bounded by the graph of the function $y = 1 + x^2$, the $x$-axis, and the lines $x = 0$ and $x = 2$.

Solution. We start from drawing graphs of all curves, and finding desire domain. This is shaded in fig. 1.

Now, at this point, we can't do this exactly. However, we can estimate the area. We will estimate the area by dividing up the interval $[0; 2]$ of $x-$axis into $n$ subintervals each of width $\Delta\ x = \frac{2}{n}$.

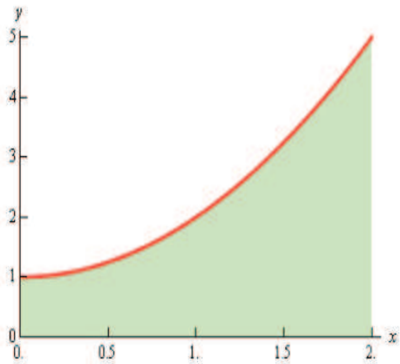


FIG. 1. Domain bounded by the graphs $y = 1 + x^2$, $y = 0$, $x = 0$ and $x = 2$

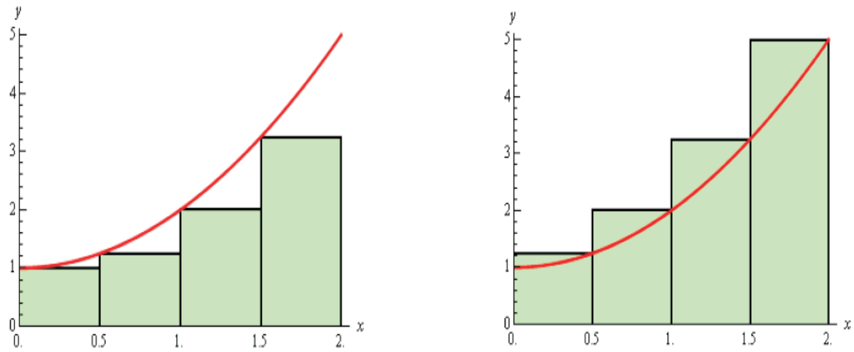


FIG. 2. Lower and upper approximations of the area

Then in each interval we can form a rectangle whose height is given by the function value at a specific point in the interval. We can then find the area of each of these rectangles, add them up and this will be an estimate of the area.

We can choose the rectangles at least in two different ways, that is shown in fig. 2. Here $n = 4$, but if we will increase the number of rectangles, the rectangles get thinner, and the approximation gets better (see fig. 3).

The left picture shows *lower approximation* and the right picture shows *upper approximation* of the desire area. Let's denote exact value of the area with $A$, the total area of rectangles shown left with $s_n$ and the total area of rectangles shown right with $S_n$. It's clear that

$$s_n < A < S_n.$$

The height of each rectangle of lower approximation over interval $[\frac{2(k-1)}{n}, \frac{2k}{n}]$ is $\left(\frac{2(k-1)}{n}\right)^2 + 1$, the height of the similar rectangle of upper approximation is $\left(\frac{2k}{n}\right)^2 + 1$, then

$$s_n = \sum_{k=1}^{n} \left( \left( \frac{2(k-1)}{n} \right)^2 + 1 \right) \frac{2}{n}; \ S_n = \sum_{k=1}^{n} \left( \left( \frac{2k}{n} \right)^2 + 1 \right) \frac{2}{n}.$$

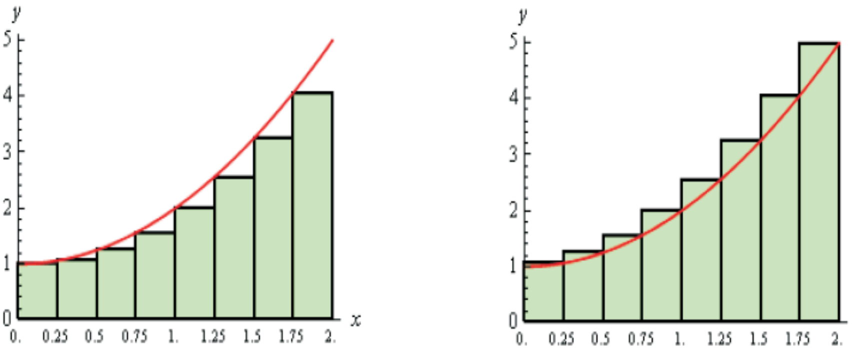


FIG. 3. Increasing number of rectangles give better approximation

Observe that, in this case, $S_n - s_n = \frac{8}{n} \underset{n\to\infty}{\longrightarrow} 0$.

Hence $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} s_n = A$.

To compute the limits observe that $\sum\limits_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$[1].

Hence $S_n = \sum\limits_{k=1}^{n} \left(\left(\frac{2k}{n}\right)^2 + 1\right)\frac{2}{n} = \frac{2}{n}\left(\frac{4}{n^2}\sum\limits_{k=1}^{n} k^2 + \sum\limits_{k=1}^{n} 1\right) = \frac{2}{n}\left(\frac{4}{n^2}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + n\right) = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} + 2 \underset{n\to\infty}{\longrightarrow} \frac{14}{3}$.

Answer: $A = \frac{14}{3}$. $\diamond$

The previous considerations were based on some intuitive idea about areas of domain. We make that precise in the statement of the following result.

Assume that $f(x)$ is continuous and $f(x) \geq 0$ for $x \in [a, b]$. Let's denote with $A$ the area of the domain under the graph of the function $f$ and over the interval $[a, b]$. To compute $A$ divide up the interval $[a; b]$ into $n$ subintervals each of width $\Delta\, x = \frac{(b-a)}{n}$. Endpoints of $k$th interval are $a + \frac{(k-1)(b-a)}{n}$, $a + \frac{k(b-a)}{n}$. Denote with $h_k(H_k)$

[1]To prove this formula verify that $\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} - \left(\frac{(n-1)^3}{3} + \frac{(n-1)^2}{2} + \frac{(n-1)}{6}\right) = n^2$.

minimal (maximal) value of function $f(x)$ on $k$th interval and put $s_n = \sum_{k=1}^{n} h_k \frac{(b-a)}{n}$, $S_n = \sum_{k=1}^{n} H_k \frac{(b-a)}{n}$.

THEOREM 2.1. $\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = A$.

DEFINITION 2.1. *The common value of limits above is called the definite integral of the function $f$ over the interval $[a, b]$ and denoted*

$$\int_a^b f(x)dx$$

.

NOTE 2.1. *It follows from the previous theorem that* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} f(x_k)\frac{(b-a)}{n} = \int_a^b f(x)dx$, *where* $x_k \in [a + \frac{(k-1)(b-a)}{n},\ a + \frac{k(b-a)}{n}]$ *is any point.*

The definite integral is defined to be exactly the limit and summation that we looked at to find the net area between a function and the $x$-axis. Due to our calculation in the example we can write

$$\int_0^2 (x^2 + 1)dx = \frac{14}{3}.$$

The number $a$ that is at the bottom of the integral sign is called *the lower limit* of the integral and the number $b$ at the top of the integral sign is called *the upper limit* of the integral. Also, despite the fact that $a$ and $b$ were given as an interval the lower limit does not necessarily need to be smaller than the upper limit. Collectively we'll often call $a$ and $b$ the interval of integration. The notation for the definite integral is very similar to the notation for an indefinite integral. The reason for this will be apparent in the next section.

Let's take a look at some of the properties of the definite integral, that follows directly from the definition.

**Properties of The Definite Integrals**

(1) $\int\limits_a^b f(x)dx = -\int\limits_b^a f(x)dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

(2) $\int\limits_a^a f(x)dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.

(3) $\int\limits_a^b kf(x)dx = k\int\limits_a^b f(x)dx$, where $k$ is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

(4) $\int\limits_a^b (f(x)+g(x))dx = \int\limits_a^b f(x)dx + \int\limits_a^b g(x)dx$. We can break up definite integrals across a sum or difference.

(5) $\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$, where $c$ is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, $[a,c]$ and $[c,b]$. Note however that $c$ doesn't need to be between $a$ and $b$.

(6) $\int\limits_a^b f(x)dx = \int\limits_a^b f(t)dt$. The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

## 2. Definite Integrals as Net Change

Previous section provides one possible interpretation of the definite integral is to give the net area between the graph of a function $f(x)$ and the the interval $[a,b]$ on $x$-axis. Here we give another interpretation, which prepares to understand the Fundamental Theorem of Calculus given in the following section.

Let $s(t)$ is the function describing the position $s$ of some object at time $t$. The displacement of the object from time $t_1$ to time $t_2$ equals $s(t_2) - s(t_1)$. From other hand, we know that the velocity of the object at any time $t$ is $v(t) = s'(t)$. For small interval of time $\Delta t$ we can consider the object move on the distance $v(t^*)\Delta t$, where $t^*$ is any time in the interval $\Delta t$. Adding all such distances and letting $\Delta t \to 0$ (so, the number of time intervals cutting out the interval $[t_1, t_2]$ tends to infinity), we obtain the definite integral $\int\limits_{t_1}^{t_2} v(x)dx$.

Therefore the displacement of the object from time $t_1$ to time $t_2$ is

$$\int\limits_{t_1}^{t_2} v(x)dx = \int\limits_{t_1}^{t_2} s'(x)dx = s(t_2) - s(t_1).$$

Note that in this case if $v(t)$ is both positive and negative (i.e. the object moves to both the right and left) in the time frame this formula will NOT give the total distance travelled. It will only give the displacement, i.e. the difference between where the object started and where it ended up. To get the total distance travelled by an object we'd have to compute $\int\limits_{t_1}^{t_2} |v(x)|dx$.

Generally, if $f(x)$ is some quantity and $f'(x)$ is the rate of change of $f(x)$, then

$$\int\limits_{a}^{b} f'(x)dx = f(b) - f(a)$$

is the net change of $f(x)$ on the interval $[a, b]$. In other words, compute the definite integral of a rate of change and you'll get the net change in the quantity.

Futher we will use the following short notation for the net change:

$$f(b) - f(a) = f(x)\Big|_a^b .$$

## 3. Definite Integral and Antiderivative. Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus consists of two parts. First part tells us how to differentiate certain types of definite integrals and it also tells us about the very close relationship between integrals and derivatives.

THEOREM 2.2 (Fundamental Theorem of Calculus, Part I). *If $f(x)$ is continuous on $[a, b]$, then $\int\limits_a^x f(t)dt$ is continious on $[a, b]$ and*

$$\left(\int\limits_a^x f(t)dt\right)' = f(x).$$

**Example:** Differentiate $g(x) = \int\limits_a^x e^{2t}\cos^2(1-5t)dt$.

Solution. According to Fundamental Theorem of Calculus, Part I,

$$g'(x) = e^{2x}\cos^2(1-5x).$$

◇

Second part of Fundamental Theorem of Calculus says how to evaluate definite integrals in practice.

THEOREM 2.3 (Fundamental Theorem of Calculus, Part II). *If $f(x)$ is continuous on $[a, b]$, and $F(x)$ is any antiderivative for $f(x)$. then*

$$\int\limits_a^b f(x)dx = F(b) - F(a).$$

Recall that when we talk about an anti-derivative for a function we are really talking about the indefinite integral for the function. So, to evaluate a definite integral the first thing that we're going to do is evaluate the indefinite integral for the function. This should explain the similarity in the notations for the indefinite and definite integrals.

**Example:** Use theorem above to compute $\int\limits_0^2 (x^2+1)dx$.

Solution. First, let's denote that $F(x) = \frac{x^3}{3} + x$ is antiderivative of $x^2+1$. Then

$$\int\limits_0^2 (x^2+1)dx = F(2) - F(0) = \left(\frac{2^3}{3}+2\right) - \left(\frac{0^3}{3}+0\right) = \frac{14}{3}.$$

◇

**Example:** Evaluate

(1) $\int (x^2+x^{-2})dx$;

(2) $\int\limits_1^2 (x^2+x^{-2})dx$;

(3) $\int\limits_{-1}^2 (x^2+x^{-2})dx$.

Solution.

(1) $\int (x^2+x^{-2})dx$. This is the only indefinite integral in this section which is here to make sure that we understand the difference between an indefinite and a definite integral. Using the table of indefinite integrals one gets

$$\int (x^2+x^{-2})dx = \frac{x^3}{3} + \frac{x^{-1}}{-1} + C = \frac{x^3}{3} - x^{-1} + C.$$

(2) $\int\limits_1^2 (x^2+x^{-2})dx$. Recall from our first example above that all we really need here is any anti-derivative of the integrand. We just computed the most general anti-derivative in the first part so we can use that if we want to. However, recall that as we noted above any constants we tack on will just cancel in the long run and so we'll use the answer from (1) without the "+c". Here's the integral

$$\int\limits_1^2 (x^2 + x^{-2})dx = \frac{x^3}{3} - x^{-1}\Big|_1^2 = \frac{2^3}{3} - 2^{-1} - \left(\frac{1^3}{3} - 1^{-1}\right) =$$

$$= \frac{2^3}{3} - 2^{-1} - \frac{1^3}{3} + 1^{-1} == \frac{17}{6}.$$

Remember that the evaluation is always done in the order of evaluation at the upper limit minus evaluation at the lower limit. Also be very careful with minus signs and parenthesis.

(3) $\int\limits_{-1}^{2} (x^2 + x^{-2})dx$. Recall that in order for us to do an integral the integrand must be continuous in the range of the limits. In this case the second term will have division by zero at $x = 0$ and since $x = 0$ is in the interval of integration, i.e. it is between the lower and upper limit, this integrand is not continuous in the interval of integration and so we can't define this integral. Note that this problem will not prevent us from doing the integral in (2) since $x = 0$ was not in the interval of integration.

◇

**Exercises:** Compute Integrals.

**2.3.1.** $\int\limits_{-3}^{1} (6x^2 - 5x + 2)dx$;

**2.3.2.** $\int\limits_{4}^{0} \sqrt{t}(t-2)dt$;

**2.3.3.** $\int\limits_{1}^{2} \sqrt{2x^5 - x + 3}x^2dx$;

**2.3.4.** $\int\limits_{-10}^{25} dx$;

**2.3.5.** $\int\limits_{0}^{1} (4x - 6\sqrt[3]{x^2})dx$;

**2.3.6.** $\int\limits_{0}^{\frac{\pi}{3}} (2\sin\theta - 5\cos\theta)d\theta$.

## 4. Substitution Rule for Definite Integrals

THEOREM 2.4. *If $g'(x)$ is continuous on $[a, b]$ and $f(x)$ is continuous on $[g(a), g(b)]$, then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt.$$

**Proof:** Let $F(x)$ be antiderivative of $f(x)$. Then, by the Fundamental Theorem of Calculus, part II,

$$\int_a^b f(g(x))g'(x)dx = F(g(b)) - F(g((a)),$$

and

$$\int_{g(a)}^{g(b)} f(t)dt = F(g(b)) - F(g(a)),$$

that give us required equality. $\diamond$

**Example:** Compute $\int_1^e \frac{\ln x}{x} dx$.

Solution. The substitution $u = \ln x$ is suggested by the function to be integrated. We have $du = \frac{1}{x}dx$; $u(1) = \ln 1 = 0$, $u(e) =\in e = 1$. Hence $\int_1^e \frac{\ln x}{x} dx = \int_0^1 udu = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$. $\diamond$

<u>**Exercises:**</u> Compute integrals below.

**2.4.1.** $\int_0^1 \sqrt{1+x}dx$;

**2.4.2.** $\int_{-2}^{-1} \frac{1}{(11+5x)^3} dx$;

**2.4.3.** $\int_0^1 (e^x - 1)^4 e^x dx$;

**2.4.4.** $\int_0^1 \frac{x}{(x^2+1)^2 dx}$;

**2.4.5.** $\int_1^e \frac{1+\ln x}{x} dx$;

**2.4.6.** $\int_2^3 \frac{dx}{2x^2+3x-2}$.

## 5. Integration by Parts for Definite Integrals

Integration by Parts Formula and the Fundamental Theorem of Calculus imply the below Integration by Parts Formula for Definite Integrals. Here we must assume that the functions $u$ and $v$ and their derivatives are all continuous.

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

<u>**Exercises:**</u> Compute integrals below.

**2.5.1.** $\int_0^1 xe^{-x}dx$;

**2.5.2.** $\int_0^{\frac{\pi}{2}} x\cos x dx$;

**2.5.3.** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{xdx}{\sin^2 x}$;

**2.5.4.** $\int_0^{\pi} x^3 \sin x dx$;

**2.5.5.** $\int_1^2 x\log_2 x dx$;

**2.5.6.** $\int_0^{e-1} \ln(x+1)dx$.

## 6. Area Between Curves

As it follows directly from the definition of the definite integral, the value of $\int_a^b f(x)dx$ can be considered as area of domain between non-negative function $f(x)$ and interval $[a,b]$.

**Example:** Find the area between the $\sin x$ curve and the $x$-axis interval $[0,\pi]$.

Solution. The area is $\int_0^{\pi} \sin x dx = -\cos x\Big|_0^{\pi} = 2.$ $\diamond$

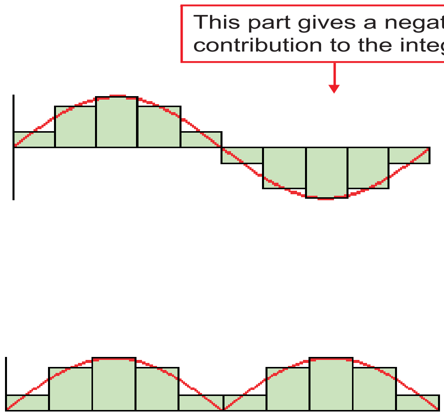


FIG. 4. The domains between the $x$-axis and the graphs of the functions $f(x)$ and $|f(x)|$ have the same area

The formula $\int_a^b f(x)dx$ for the area can be applied only if the function $f$ does not take negative values. Case when $f(x)$ takes negative values is illustrated in the fig. 4 and in the following example.

Hence if the function f takes also negative values, then the area of the domain between the graph of $f$ and the $x$-axis is given by the integral

$$\int_a^b |f(x)|dx.$$

**Example:** Find the area between the $\sin x$ curve and the $x$-axis interval $[0, 2\pi]$.

Solution. The area is $\int_0^{2\pi} |\sin x|dx = \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} (-\sin x)dx = 4.$

◇

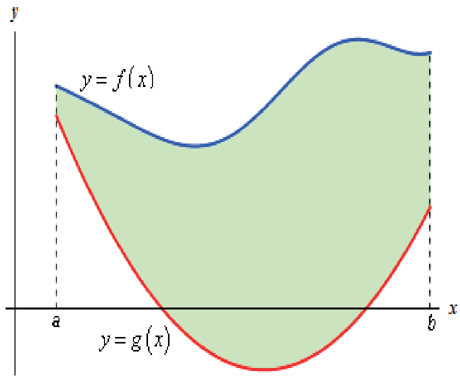


FIG. 5. Area between graphs of the functions $f(x)$ and $g(x)$

Extending the same idea one can get formula for computing area between two curves, defined by functions $f(x)$, $g(x)$ (see fig. 5).

$$\boxed{A = \int_a^b |f(x) - g(x)| dx}$$

**Example:** Determine the area of the region enclosed by $y = \sqrt{x}$ and $y = x^2$.

Solution. First of all, just what do we mean by "area enclosed by". This means that the region we're interested in must have one of the two curves on every boundary of the region. So, fig. 6 presents graphs of the two functions with the enclosed region shaded.

We should find points of intersection of the graphs to get limits of integrations. When $y = \sqrt{x}$ meets $y = x^2$

$$\sqrt{x} = x^2,$$

and $x = 0,\ x = 1$ are the solutions of this equation.

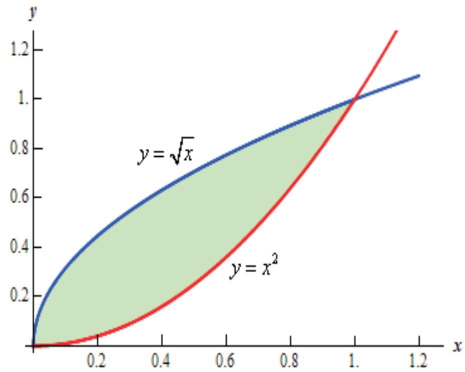


FIG. 6. Area between graphs of the functions $f(x) = \sqrt{x}$ and $g(x) = x^2$

The integral we need to compute to find the area is

$$\int_0^1 (\sqrt{x} - x^2)dx = \frac{1}{3}.$$

$\diamond$

**Exercises:**

**2.6.1.** Determine the area of the region enclosed by $y = 2x^2 + 10$ and $y = 4x + 16$.

**2.6.2.** Determine the area of the region bounded by $y = xe^{-x^2}$, $y = x + 1$, $x = 2$ and $y$-axis.

**2.6.3.** Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$.

**2.6.4.** Determine the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$ and $y$-axis.

**2.6.5.** Determine the area of the region bounded by $x = \frac{1}{2}y^2 - 3$, $y = x - 1$.

**2.6.6.** Determine the area of the region bounded by $x = -y^2 + 10$, $yx = (y - 2)^2$.

## Samples of Test Cards

**Card 1**

1) Find integrals:
   a) $\int xe^{-x^2}dx$
   b) $\int x\cos(2x)dx$
2) Find the area of domain bounded by lines $y = \frac{1}{x-1}$, $y = 0$, $x = 2$, $x = 3$.

**Card 2**

1) Find integrals:
   a) $\int \sin(2x)dx$
   b) $\int x\ln x dx$
2) Find the area of domain bounded by lines $y = \sqrt{x}$, $y = 0$, $x = 4$, $x = 9$.

**Card 3**

1) Find integrals:
   a) $\int \frac{\ln x}{x}dx$
   b) $\int xe^{3x}dx$
2) Find the area of domain bounded by lines $y = x^3$, $y = 0$, $x = 2$.

**Card 4**

1) Find integrals:
   a) $\int \frac{e^{\frac{1}{x}}}{x^2}dx$
   b) $\int \arccos x dx$
2) Find the area of domain bounded by lines $y = 1 - x^2$, $y = 0$.

## Application: Table of derivatives and Rules of Differentiation

| | |
|---|---|
| $(x^n)' = nx^{n-1}$ | $(\ln x)' = \frac{1}{x}$ |
| $(e^x)' = e^x$ | $(a^x)' = a^x \ln a$ |
| $(\sin x)' = \cos x$ | $(\cos x)' = -\sin x$ |
| $(\tan x)' = \frac{1}{\cos^2 x}$ | $(\cot x)' = -\frac{1}{\sin^2 x}$ |
| $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ | $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ |
| $(\arctan x)' = \frac{1}{1+x^2}$ | $(\mathrm{arccot} x)' = -\frac{1}{1+x^2}$ |
| $(\sec(x))' = \sec x \tan x$ | $(\mathrm{cosec} x)' = -\operatorname{cosec} x \cot x$ |
| $(\mathrm{arcsec} x)' = \frac{1}{x\sqrt{x^2-1}}$ | $(\mathrm{arccosec} x)' = -\frac{1}{x\sqrt{x^2-1}}$ |

## Linearity of Derivative

$$(ky(x))' = ky'(x) \text{ where } k \text{ is any number}$$

$$(f(x) + g(x))' = f'(x) + g'(x)$$

## Product and Division Rules

$$(uv)' = u'v + uv'$$
$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

## Chain Rule

$$(f(g(x)))' = f'(g(x))g'(x)$$

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## Методы интегрирования в примерах и задачах

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