МИНИСТЕРСТВО ОБРАЗОВАНИЯ И НАУКИ РФ

Нижегородский государственный университет им. Н.И. Лобачевского

С.В. Сорочан

Основы дискретной математики

Учебно-методическое пособие

Рекомендовано методической комиссией факультета иностранных студентов для англоязычных иностранных студентов ННГУ, обучающихся по направлению подготовки 010400 — «Фундаментальная информатика и информационные технологии».

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Рецензент:

Настоящее пособие содержит англоязычные материалы по основам дискретной математики. Предлагается адаптированный вариант курса “Основы дискретной математики”, включающий в себя краткие конспекты лекций. Также излагается тематика практических занятий и приводятся задания для самостоятельной работы и вопросы к экзамену.

Учебно-методическое пособие предназначено для англо-говорящих иностранных студентов младших курсов, специализирующихся по направлению подготовки 010400 — «Фундаментальная информатика и информационные технологии».

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MINISTRY OF EDUCATION AND SCIENCE OF RUSSIAN FEDERATION

N.I. Lobachevsky State University of Nizhny Novgorod

S.V. Sorochan

Fundamentals of Discrete Mathematics

Studying-methodical manual

This manual is recommended by Methodical Committee of the Department of Foreign Students for English-speaking students of Nizhny Novgorod State University studying at Bachelor's Program 010400 — «Fundamental Informatics and Information Technologies».

1-st edition

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2011

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С-10 Sorochan S.V. FUNDAMENTALS OF DISCRETE MATHEMATICS: Studying manual. — Nizhny Novgorod: State University of Nizhny Novgorod, 2011.

Reviewer:

This manual contains materials in English on fundamentals of Discrete Mathematics. The adapted variant of the course "Fundamentals of Discrete Mathematics" including abstracts of lectures is offered. Also the topics of practical classes are described, problems for independent work and examination questions are given.

The studying-methodical manual is recommended for English-speaking foreign students of the 1-st and 2-nd years specializing at Bachelor's Program 010400 — «Fundamental Informatics and Information Technologies».

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Chapter 2. Binary relations.

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Chapter 3. Elements of combinatorics.

Basic principles of counting: the sum rule and the product rule. Permutations and combinations, their number, properties of combination numbers. Newtonian Binomial. Partitions with given specification. Inclusion-exclusion method.

Chapter 4. Recurrence equations.

First order linear recurrences, their solving. Second order homogeneous linear recurrence, its characteristic equation. Second order inhomogeneous linear recurrence, its reduction to homogeneous case.

Chapter 1. Sets and set operations.

§1. Notion of a set.

A ***set*** is a collection of distinct objects.

Members of a set are called ***elements***.

The order of elements in a set does not matter.

Mathematicians consider sets consisting of mathematical objects (numbers, functions, points, lines etc.)

A set may be ***finite*** or ***infinite***. Any finite set may be specified by listing its elements. Elements are listed between braces:

$A = \{1, 2, 4, 8\}$,

$B = \{x, y, z\}$,

$C = \{red, yellow, green\}$,

$D = \{Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday\}$.

Denotations:

$x \in A$ means "element x belongs to the set A",

$x \notin A$ means "element x does not belong to the set A".

For instance,

$2 \in A$, $z \in B$, $Thursday \in D$,

$7 \notin A$, $w \notin B$, $white \notin C$.

Empty set is the set having no elements. It is denoted by $\varnothing$.

A ***cardinality*** (or a ***size***) of a finite set X is the number of elements in X. Cardinality of the set X is denoted by $|X|$.

Examples:

$|\{x, y, z\}| = 3$, $|\{-2, -1, 0, 1, 2\}| = 5$, $|\varnothing| = 0$, $|\{\varnothing\}| = 1$.

Examples of infinite sets:

$\mathbb{N}$ is the set of all naturals, its elements are 1, 2, 3, …

$\mathbb{N}_0$ is $\mathbb{N}$ with added element 0, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Z}$ is the set of all integers, its elements are 0, 1, –1, 2, –2, 3, -3 …

$\mathbb{Q}$ is the set of all rationals, $\mathbb{Q} = \{ p/q : p \in \mathbb{Z}, q \in \mathbb{N}\}$.

$\mathbb{R}$ is the set of all real numbers.

The set of all points in a plane.

§2. Subsets. Characteristic vector of a subset.

A set A is called a ***subset*** of a set B if every element of A belongs to B.

The record $A \subseteq B$ means "A is subset of B",

one says also "A is included to B" or "B includes A".

A subset of size k is also called a ***k-subset***.

Examples:

$\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \subseteq \mathbb{R}$, $\{2, 4\} \subseteq \{1, 2, 3, 4\}$,

$\{x \in \mathbb{R}:\ 5 \leqslant x < 6\} \subseteq \{x \in \mathbb{R}:\ x > 4\}$.

For any sets A, B and C the following properties take place:

$\varnothing \subseteq A$,

$A \subseteq A$,

if $A \subseteq B$ and $B \subseteq A$ then $A = B$,

if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Sometimes it is convenient to suppose that there is a ***universal set*** (***universe***) U that contains all elements under consideration. For instance, if we study properties of naturals then $U = \mathbb{N}$. If we deal with geometric objects then we take U as the set of all points in a plane.

Often a set is specified by a property P that selects elements of this set from a universe U in the following form:

$$\{x \in U:\ P(x)\} \quad \text{or} \quad \{x:\ x \in U \text{ and } P(x)\}.$$

Here $P(x)$ means "element x has property P".

Examples:

$\{x \in \mathbb{N}:\ x \text{ is even}\}$ is the set of all even naturals,

$\{x \in \mathbb{Z}: x > 0\} = \mathbb{N}$

$\{x \in \mathbb{R}:\ 1 < x < 4\}$ is an interval in the set of all reals,

$\{\, x:\ x \in \mathbb{R} \text{ and } x^2 - 2 = 0\,\} = \{\, \sqrt{2}, -\sqrt{2}\,\}$ is finite set having two elements.

Let U be finite universe which elements are indexed by numbers $1, 2, \ldots, n$:

$$U = \{u_1, u_2, \ldots, u_n\}.$$

A subset A of U can be specified with a help of a sequence

$$h(A) = (h_1, h_2, \ldots, h_n)$$

$$\text{where} \quad h_i = 1 \text{ if } u_i \in A, \quad h_i = 0 \text{ if } u_i \notin A, \quad i = 1, 2, \ldots, n.$$

This sequence is called ***characteristic vector*** of set A.

Examples. $U = \{a, b, c, d, e\}$,

$A = \{a, c, e\} \Rightarrow h(A) = (1, 0, 1, 0, 1)$,

$B = \{d, e\} \Rightarrow h(B) = (0, 0, 0, 1, 1)$,

$C = \{a\} \Rightarrow h(C) = (1, 0, 0, 0, 0)$,

$h(\varnothing) = (0, 0, 0, 0, 0)$, $h(U) = (1, 1, 1, 1, 1)$.

§3. Set operations.

Union of two sets A and B is defined as the set

$A \cup B = \{\, x:\ x \in A \text{ or } x \in B\}$.

Example:

$A = \{0, 1, 4\}, \quad B = \{1, 2, 4\},$

$A \cup B = \{0, 1, 2, 4\}.$

Intersection of A and B is defined as the set

$A \cap B = \{ x : x \in A \text{ and } x \in B\}.$

Example:

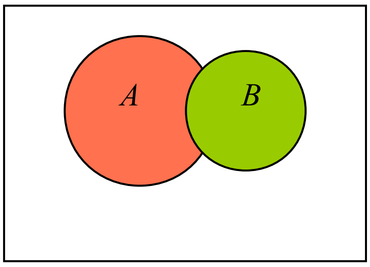
$A = \{0, 1, 4\}, \quad B = \{1, 2, 4\},$

$A \cap B = \{1, 4\}.$

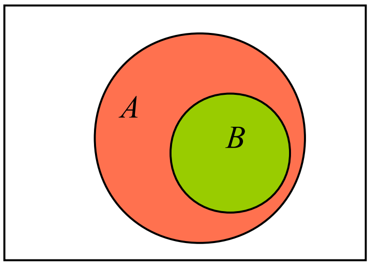
Two sets A and B are called ***disjoint*** if $A \cap B = \varnothing$.

Venn diagram is a graphical way for representation of relations between sets. We draw a universe as a rectangle and its subsets as circles or other figures in this rectangle.

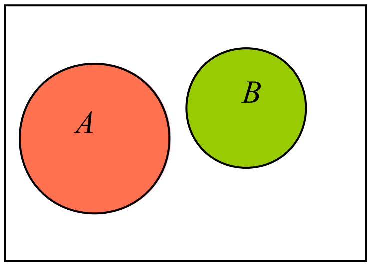
Examples:



Sets A and B intersect, i.e. $A \cap B \neq \varnothing$

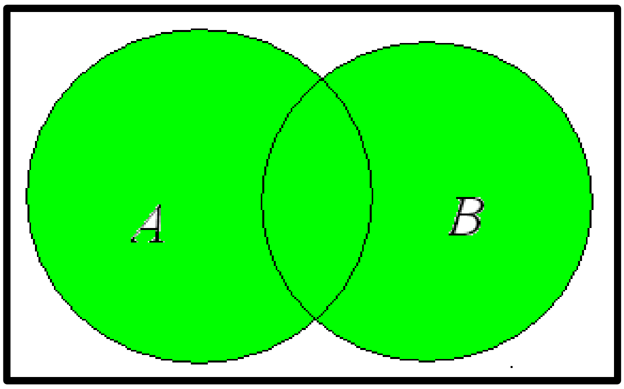


B is subset of A, i.e. $B \subseteq A$

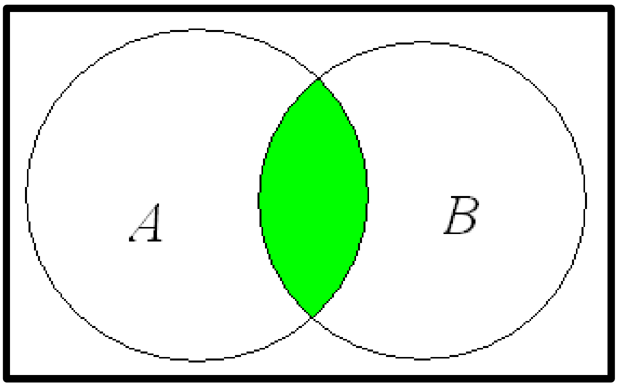


Sets A and B are disjoint, i.e. $A \cap B = \varnothing$

Graphical representations of union and intersection operations:



$A \cup B$



$A \cap B$

Properties of union and intersection operations:

$A \cup A = A, \quad A \cap A = A, \quad A \cup \varnothing = A,$

$A \cap \varnothing = \varnothing, \quad A \cup U = U, \quad A \cap U = A$ for any set A.

Both operations are ***commutative***:

$A \cup B = B \cup A$,

$A \cap B = B \cap A$

for any sets A, B.

Both operations are ***associative***:

$A \cup (B \cup C) = (A \cup B) \cup C$,

$A \cap (B \cap C) = (A \cap B) \cap C$

for any sets A, B.

Due to associativity it's reasonable to omit parentheses and write simply

$A \cup B \cup C, \quad A \cap B \cap C.$

If there are n sets $A_1, A_2, \dots, A_n$ then we write

$A_1 \cup A_2 \cup \dots \cup A_n, \quad A_1 \cap A_2 \cap \dots \cap A_n.$

Union and intersection operations are linked by the following distributive laws:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

for any sets A, B, C.

Using these distributive laws one can deduce the following ***absorption laws***:

$A \cup (A \cap B) = A$,

$A \cap (A \cup B) = A$ for any sets A and B.

Difference of sets A and B is the set

$A - B = \{x: \ x \in A \text{ and } x \notin B\}$.

Example:

$A = \{0, 1, 4\}, \quad B = \{1, 2, 4\}$,

$A - B = \{0\}$.

Complement of a set A with respect to a universe U is the set

$\overline{A} = U - A$.

Example:

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, \quad A = \{0, 2, 4, 6, 8\}$,

$\overline{A} = \{1, 3, 5, 7, 9\}$.

Union, intersection and complement operations are linked by two De Morgan laws:

$\overline{A \cup B} = \overline{A} \cap \overline{B}$,

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

for any sets A, B.

The following identities are true for any sets A and B:

$\overline{\overline{A}} = A, \quad A \cap \overline{A} = \varnothing, \quad A \cup \overline{A} = U, \quad A - B = A \cap \overline{B}.$

Symmetric difference of sets A and B is the set

$A \otimes B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Example:

$A = \{0, 1, 4\}$, $B = \{1, 2, 4\}$,

$A \otimes B = \{0\} \cup \{2\} = \{0, 1, 2, 4\} - \{1, 4\} = \{0, 2\}$.

The inclusion and equality of sets can be expressed in terms of set operations:

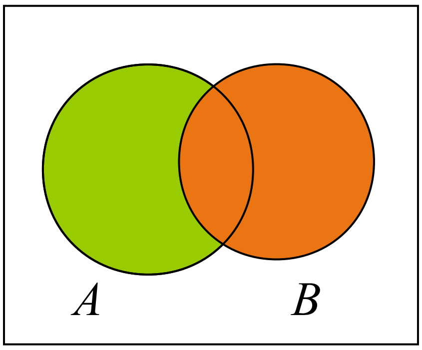
$A \subseteq B$ if and only if $A \cap B = A$,

$A \subseteq B$ if and only if $A \cup B = B$,

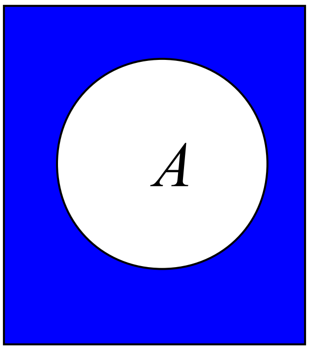
$A \subseteq B$ if and only if $A - B = \varnothing$,

$A = B$ if and only if $A \otimes B = \varnothing$

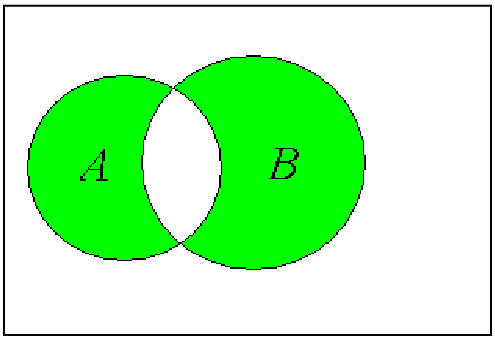
for any sets A and B.



$A - B$



$\overline{A}$



$A \otimes B$

The inclusion and equality of sets can be expressed in terms of set operations:

$A \subseteq B$ if and only if $A \cap B = A$,

$A \subseteq B$ if and only if $A \cup B = B$,

$A \subseteq B$ if and only if $A - B = \varnothing$,

$A = B$ if and only if $A \otimes B = \varnothing$

for any sets A and B.

§4. Set equations.

Set equation is an equality containing several known and unknown sets. To solve such an equation means to describe all possible values of unknown sets for which the equality becomes identity.

Set equations having one unknown set X are considered their. We suppose that all considered sets are subsets of some universe U.

General method for solving an arbitrary set equation of this kind is offered below.

Further to simplify expressions having several operations of union and intersection we shall write AB instead of $A \cap B$ and assume that the operation $\cap$ is stronger than $\cup$. It means that $\cap$ is executed before $\cup$ if there are no parenthesis defining another order of operations. For example, the expression $AB \cup CD$ is equivalent to $(A \cup B) \cup (C \cup D)$.

First of all we solve the following two simplest equations.

Example 1.

Solve set equation $A\overline{X} = \varnothing$, where X is unknown set, A is given set.

Solution:

$A\overline{X} = \varnothing \Leftrightarrow A - X = \varnothing \Leftrightarrow A \subseteq X$ (the sign $\Leftrightarrow$ means "if and only if").

Example 2.

Solve set equation $B\,X = \varnothing$, where X is unknown set, B is given set.

Solution:

$B\,X = \varnothing \Leftrightarrow X - \overline{B} = \varnothing \Leftrightarrow X \subseteq \overline{B}$.

Now we demonstrate general method for solving set equations on several examples.

Example 3. Solve set equation

$A \cup X = B$, where X is unknown set, A and B are given sets.

Solution:

$A \cup X = B$.

Two sets are equal if and only if their symmetric difference is the empty set. Hence the given equation is equivalent to the equation

$(A \cup X) \otimes B = \varnothing$.

If we transform the left side using the identity

$S \otimes B = (S - B) \cup (B - S) = S\overline{B} - B\overline{S}$ (where $S = A \cup X$) then we get

$(A \cup X)\overline{B} \cup \overline{(A \cup X)}B = \varnothing$.

Now we remove parentheses with a help of distributive law and use De Morgan law (our goal is to obtain an expression without parentheses where every complement is taken of single set):

$A\overline{B} \cup X\overline{B} \cup \overline{A}\,\overline{X}B = \varnothing$.

The union of several sets is empty if and only if each of them is empty. Thus, the last equation is equivalent to the following system of three equations:

$$\begin{cases} A\overline{B} = \varnothing, \\ X\overline{B} = \varnothing, \\ \overline{A}B\overline{X} = \varnothing. \end{cases}$$

The first of these equations does not contain unknown set. It represents necessary condition for existence of a solution. It is equivalent to the inclusion $A \subseteq B$.

The second and the third equations are equations of the simplest type (see examples 1 and 2). Their solutions are

$$X \subseteq B \quad \text{and} \quad \overline{A}\, B = B - A \subseteq X, \text{ correspondently.}$$

So, we conclude that initial equation has a solution if and only if

$$A \subseteq B \quad \text{(it's necessary condition)}$$

and provided this condition any set X satisfying the inclusions

$$B - A \subseteq X \subseteq B$$

is the solution of the equation.

There is a way to verify the obtained solution. For this purpose the so-called parameterized form of a solution is used.

It's difficult to verify a solution written in the form of inclusion chain. ***Parameterized form*** of a solution is an equality where unknown set is represented as a set depending on some arbitrary subset P of a universe named a ***parameter***.

For our example it's not difficult to see that the chain of inclusions

$$B - A \subseteq X \subseteq B$$

is equivalent to the equality

$$X = \overline{A}\, B \cup PA, \quad P \subseteq U.$$

It remains to substitute unknown set X in the initial equation using this parameterized form and verify that the equality takes place as identity:

$$A \cup X = A \cup \overline{A}\, B \cup P\, A = (A \cup PA) \cup \overline{A}\, B = A \cup \overline{A}\, B = (A \cup \overline{A})\, (A \cup B) =$$
$$= U\, (A \cup B) = A \cup B = B \quad \text{because} \quad A \subseteq B$$

(we used absorption laws, distributive law and necessary condition), q.e.d.

Example 4.

Solve set equation

$A\, X = B$, where X is unknown set, A and B are given sets.

Solution:

$$AX = B \Leftrightarrow AX \otimes B = \varnothing \Leftrightarrow AX\,\overline{B} \cup B\,\overline{AX} = \varnothing \Leftrightarrow$$
$$\Leftrightarrow AX\,\overline{B} \cup B\,(\overline{A} \cup \overline{X}) = \varnothing \Leftrightarrow AX\,\overline{B} \cup B\,\overline{A} \cup B\,\overline{X} = \varnothing \Leftrightarrow$$

$$\begin{cases} A\,X\,\overline{B} = \varnothing, \\ B\,\overline{A} = \varnothing, \\ B\,\overline{X} = \varnothing. \end{cases}$$

The latter system has the solution

$$B \subseteq X \subseteq \overline{A\,\overline{B}} = \overline{A} \cup B \quad \text{provided necessary condition} \quad B \subseteq A.$$

Verification.

We write the solution in parameterized form and substitute it to the initial equation:

$$X = B \cup P\overline{A}, \qquad P \subseteq U.$$

$A X = A(B \cup P\overline{A}) = AB \cup AP\overline{A} = AB \cup \varnothing = AB = B$ (because $B \subseteq A$), q.e.d.

Example 5. Solve set equation

$A - X = X - B$, where X is unknown set, A and B are given sets.

Solution:

$$A\overline{X} = X\overline{B} \Leftrightarrow A\overline{X} \otimes X\overline{B} = \varnothing \Leftrightarrow (A\overline{X} - X\overline{B}) \cup (X\overline{B} - A\overline{X}) = \varnothing \Leftrightarrow$$

$$\Leftrightarrow A\overline{X}\,\overline{X\overline{B}} \cup X\overline{B}\,\overline{A\overline{X}} = \varnothing \Leftrightarrow A\overline{X}(\overline{X} \cup B) \cup X\overline{B}(\overline{A} \cup X) = \varnothing \Leftrightarrow$$

$$\Leftrightarrow A\overline{X} \cup A\overline{X}B \cup X\overline{B}\,\overline{A} \cup X\overline{B} = \varnothing \Leftrightarrow A\overline{X} \cup X\overline{B} = \varnothing \Leftrightarrow$$

$$\begin{cases} A\overline{X} = \varnothing, \\ X\overline{B} = \varnothing. \end{cases}$$

The latter system has the solution $A \subseteq X \subseteq B$
and from the inclusions it follows that there exists necessary condition $A \subseteq B$.

Verification.

We write the solution in parameterized form and substitute it to the initial equation:

$$X = A \cup P(B - A) = A \cup PB\overline{A}, \qquad P \subseteq U.$$

$A - X = A\overline{X} = A(\overline{A \cup PB\overline{A}}) = A\overline{A}\;\overline{PB\overline{A}} = \varnothing,$

$X - B = (A \cup PB\overline{A})\overline{B} = A\overline{B} \cup PB\overline{A}\,\overline{B} = A\overline{B} \cup \varnothing = \varnothing$ (because $A \subseteq B$), q.e.d.

§5. Power set.

Element of a set can form a set itself. For example, the set

$$X = \{\{a, b\}, \{a\}, \varnothing, \{b, c\}, \{a, b, c\}\}$$

contains 5 elements.

If the elements of a considered set X are subsets of a set A then one calls X a ***family*** of subsets. The above set X is a family of subsets for the set $A = \{a, b, c\}$.

The family of all subsets for a set A is denoted by 2^A. It is called ***power set***. How many elements does the set 2^A have?

If $A = \varnothing$ then there is only subset $\varnothing$.
If $A = \{a\}$ then there are 2 subsets: $\varnothing$, $\{a\}$.
If $A = \{a, b\}$ then there are 4 subsets: $\varnothing$, $\{a\}$, $\{b\}$, $\{a, b\}$.
If $A = \{a, b, c\}$ then there are 8 subsets:

$$\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

Theorem. If $|A| = n$ then $|2^A| = 2^n$.

Example. Find power set for the set $A = \{1, 4, 5, 9\}$.

Solution:

$$2^A = \{\varnothing, \{1\}, \{4\}, \{5\}, \{9\}, \{1, 4\}, \{1, 5\}, \{1, 9\}, \{4, 5\}, \{4, 9\}, \{5, 9\},$$
$$\{1, 4, 5\}, \{1, 4, 9\}, \{1, 5, 9\}, \{4, 5, 9\}, \{1, 4, 5, 9\}\}.$$

§6. Cartesian product of sets.

Sequence is a collection of elements (finite or infinite) written in certain order. When writing the sequence we use the parentheses. For instance,

$(a_1, a_2, \ldots, a_n)$ is a sequence of length n.

There are two distinctions between the concepts of a sequence and a set.

1. The order of elements in a sequence is essential:
   (a, b, c) and (b, c, a) are different sequences but
   $\{a, b, c\}$ is the same set as $\{b, c, a\}$.
2. An element can occur in a sequence more then once:
   $(1, 2, 1, 3, 2, 1)$ is a sequence consisting of elements belonging to the set $\{1, 2, 3\}$.

A sequence of length 2 is called a *pair*.

Cartesian product (also named ***direct product***) of two sets A and B is the set of all pairs (a, b) where $a \in A, b \in B$. In symbols:

$$A \times B = \{(a, b)\colon\ a \in A, b \in B\}.$$

If $A = B$ then we get the set $A \times A = A^2$. It is called ***Cartesian square*** of A.

Examples.

$A = \{a, b, c\}$, $B = \{0,1\}$.

$A \times B = \{(a, 0), (a, 1), (b, 0), (b, 1), (c, 0), (c, 1)\}$,

$A^2 = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$,

$B^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

If we have n sets $A_1, A_2, \ldots, A_n$ then their Cartesian product is

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n)\colon\ a_1 \in A_1,\ a_2 \in A_2, \ldots, a_n \in A_n\}.$$

If $A_1 = A_2 = \ldots = A_n = A$ then the product is called ***n-th Cartesian degree*** and is denoted by A^n.

Example:

$A = \{0,1\}$.

$A^3 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$.

Chapter 2. Binary relations.

§1. Notion of binary relation. Ways of its representation.

Let A and B be sets. A ***relation*** R between A and B is a subset of the set $A \times B$ (Cartesian product of sets A and B): $R \subseteq A \times B$.

If $A = B$ then $R \subseteq A^2$ is relation on the set A.

If R is relation and $(x, y) \in R$ then one says "element x is in the relation R to element y" and writes $x\,R\,y$.

Matrix representation of relations.

A relation between finite sets can be represented by ***matrix*** (rectangular array). Let R be a relation between sets A and B. Matrix of the relation is the array M which rows correspond to elements of A and columns correspond to elements of B. In the cell corresponding to elements $x \in A$, $y \in B$ we write the element $m(x, y)$ of M defined in the following way:

$$m(x, y) = 1 \text{ if } (x, y) \in R,$$
$$m(x, y) = 0 \text{ if } (x, y) \notin R.$$

Example 1.

Let A = {*signal lights, tiger's skin, cucumber, water-melon*},

B = {*red, yellow, green*},

and let relation R between A and B be defined as follows:

$x\,R\,y$ if and only if color y appears in object x.

If we use the abbreviations:

$A = \{s, t, c, w\}$,

$B = \{r, y, g\}$

then formally this relation may be specified as the set of pairs

$$R = \{(s,r), (s,y), (s,g), (t,y), (c,g), (w,g), (w,r)\}.$$

Matrix M of this relation is

M	*r*	*y*	*g*
s	1	1	1
t	0	1	0
c	0	0	1
w	1	0	1

If R is a relation on a set A then its matrix M is square.

Example.

Let R be ***divisibility*** relation on the set $A = \{1, 2, 3, 4, 5, 6\}$. It is defined in the following way:

element $x \in A$ divides element $y \in A$ if there exists such $k \in A$ that $x\,k = y$.

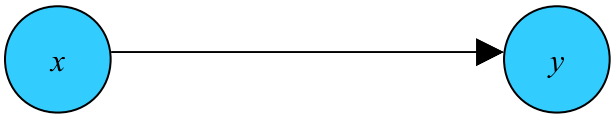
If x divides y then we write $x \mid y$.

Matrix M of this relation is

M	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	0	1	0	1
3	0	0	1	0	0	1
4	0	0	0	1	0	0
5	0	0	0	0	0	1
6	0	0	0	0	0	1

Graphical representation of relations.

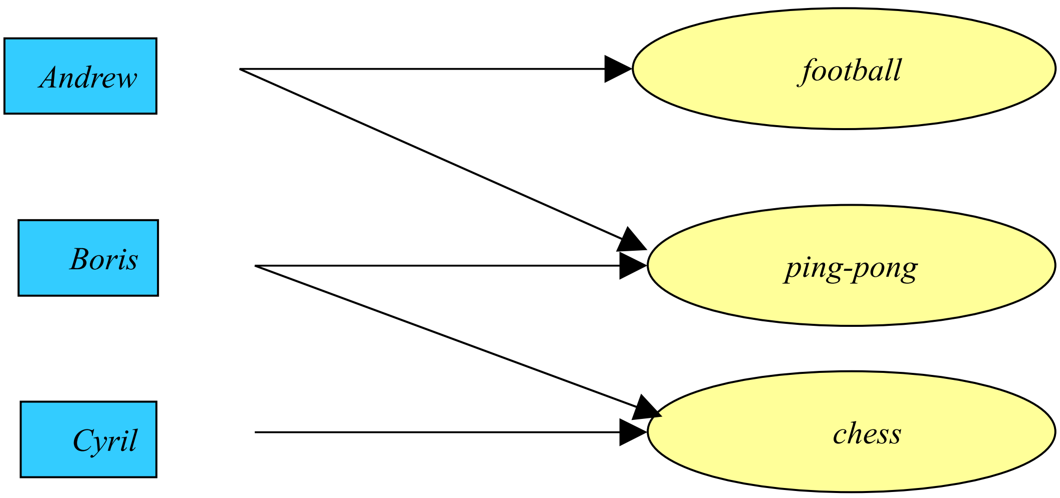
Graph of a relation gives its visual representation. This graph is constructed in the following way. Let R be a relation between sets A and B. Elements of the set $A \cup B$ are represented by circles or by rectangles or by triangles or by some other figures. These figures are named ***vertices*** of the graph. If $x\,R\,y$ holds then we draw an arrow from x to y:



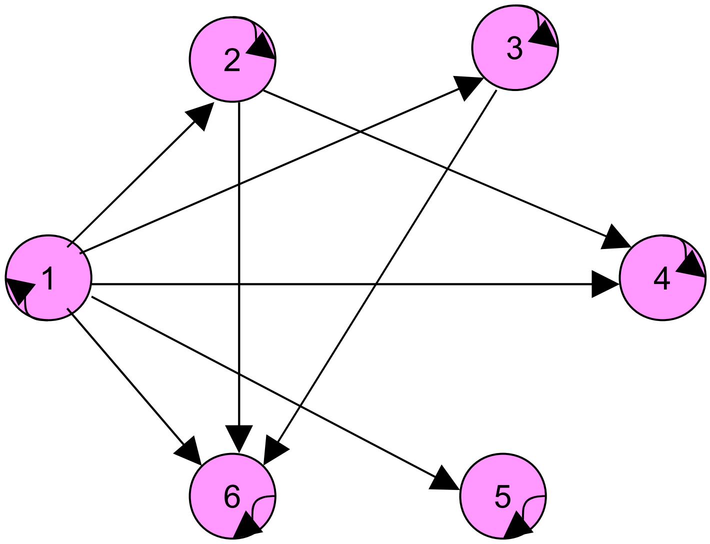
These arrows are called ***edges*** of the graph.

Examples.

1. Suppose there is a set of three students A = {*Andrew, Boris, Cyril*}. Each of the students likes some kinds of sport from the set B = {*football, ping-pong, chess*}. Namely, Andrew likes football and ping-pong, Boris likes ping-pong and chess, Cyril likes only chess. The relation between A and B can be represented by the following graph:



2. There is the graph of divisibility relation given on the set $A = \{1, 2, 3, 4, 5, 6\}$ (see the definition of divisibility relation above):



§2. Operations with relations.

Any a relation is a set (of pairs). So, we can apply any set operation to relations.

If R_1 and R_2 are relations between sets A and B (or relation on a set A) then $R_1 \cup R_2$, $R_1 \cap R_2$ etc. are also relations between A and B (on A).

Analogously, if R is a relation between A and B (on A) then $\overline{R}$ is the complement of R with respect to $A \times B$ (to A^2).

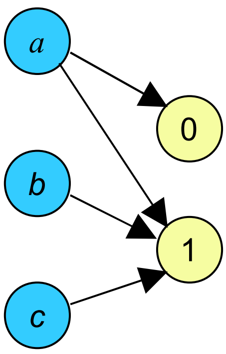
Example 1.

Let $A = \{a, b, c\}$, $B = \{0, 1\}$ and

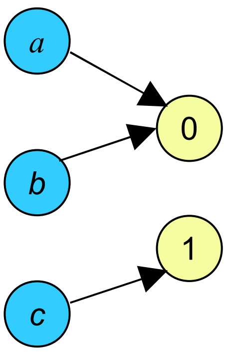
$R_1 = \{(a, 0), (a, 1), (b, 1), (c, 1)\}$, $R_2 = \{(a, 0), (b, 0), (c, 1)\}$

be relations between the sets A and B. Then

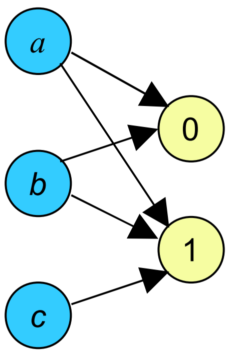
$R_1 \cup R_2 = \{(a, 0), (a, 1), (b, 0), (b, 1), (c, 1)\}$, $R_1 \cap R_2 = \{(a, 0), (c, 1)\}$.



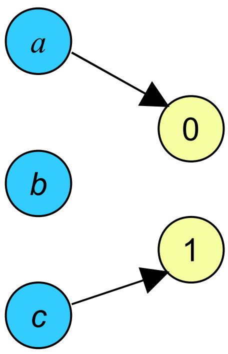
R_1



R_2



$R_1 \cup R_2$



$R_1 \cap R_2$

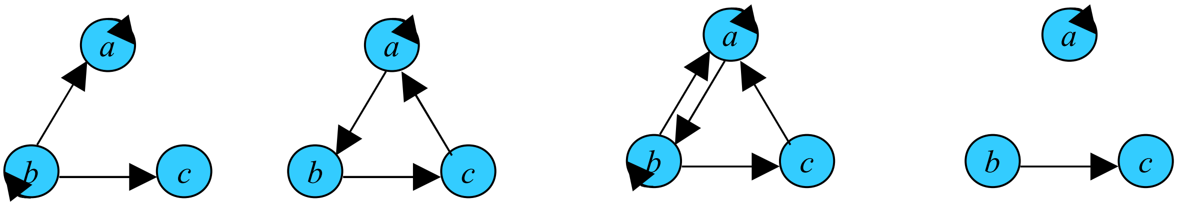
Example 2.

Let $A = \{a, b, c\}$ and

$R_1 = \{(a, a), (b, a), (b, b), (b, c)\}$, $R_2 = \{(a, a), (a, b), (b, c), (c, a)\}$

be relations on the set A. Then

$R_1 \cup R_2 = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, a)\}$, $R_1 \cap R_2 = \{(a, a), (b, c)\}$.



R_1 R_2 $R_1 \cup R_2$ $R_1 \cap R_2$

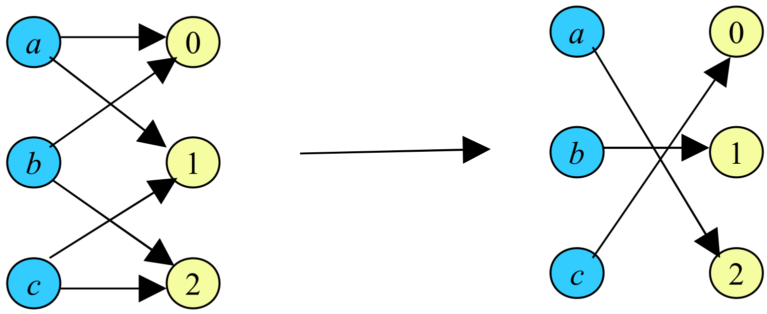
Example 3.

Let $A = \{a, b, c\}$, $B = \{0, 1, 2\}$ and

$R = \{(a, 0), (a, 1), (b, 0), (b, 2), (c, 1), (c, 2)\}$

be relation between the sets A and B. Then

$\overline{R} = \{(a, 2), (b, 1), (c, 0)\}$.



R $\overline{R}$

Example 4.

Let $A = \{a, b, c, d\}$ and

$R = \{(a, a), (a, b), (b, b), (b, d), (c, b), (d, a), (d, b), (d, c)\}$

be relation on the set A. Then

$\overline{R} = \{(a, c), (a, d), (b, a), (b, c), (c, a), (c, c), (c, d), (d, d)\}$.

R $\overline{R}$

Let R be a relation between sets A and B (or relation on a set A).

Inverse relation R^{-1} is defined as

$R^{-1} = \{(x, y): (y, x) \in R\}$.

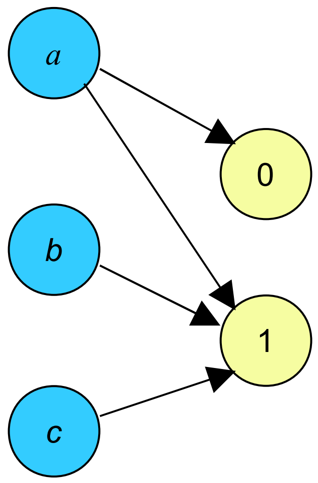
If R is a relation between A and B (on A) then R^{-1} is a relation between B and A (on A):

$$R \subseteq A \times B \Rightarrow R^{-1} \subseteq B \times A; \qquad R \subseteq A^2 \Rightarrow R^{-1} \subseteq A^2.$$

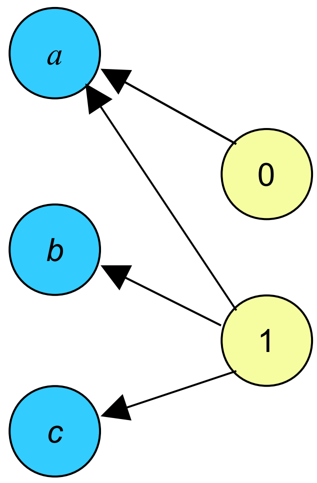
Example 5.

$A = \{a, b, c\}$, $B = \{0, 1\}$, $R = \{(a, 0), (a, 1), (b, 1), (c, 1)\} \subseteq A \times B \Rightarrow$

$R^{-1} = \{(0, a), (1, a), (1, b), (1, c)\} \subseteq B \times A$.



R

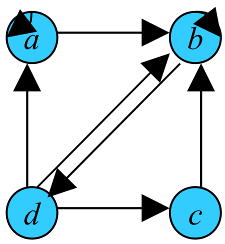


R^{-1}

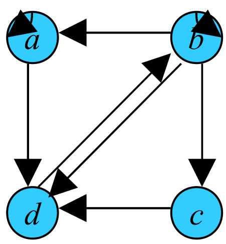
Example 6.

$A = \{a, b, c, d\}$, $R = \{(a, a), (a, b), (b, b), (b, d), (c, b), (d, a), (d, b), (d, c)\} \subseteq A^2$

$\Rightarrow R^{-1} = \{(a, a), (b, a), (b, b), (d, b), (b, c), (a, d), (b, d), (c, d)\} \subseteq A^2$.



R



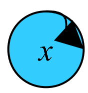
R^{-1}

§3. Properties of relations.

Let R be a relation on a set A ($R \subseteq A^2$).

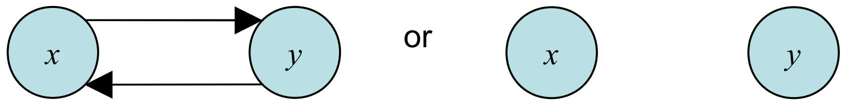
R is called ***reflexive*** if xRx ($(x, x) \in R$) holds for any element $x \in A$.

For any reflexive relation main diagonal of its matrix contains ones only. In a graph of reflexive relation every vertex has a *loop*:



Relation R is called ***symmetric*** if xRy implies yRx for all $x, y \in A$ $((x, y) \in R \rightarrow (y, x) \in R)$.

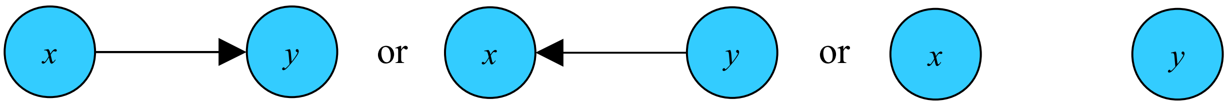
Matrix of any symmetric relation is symmetric with respect to its main diagonal. In the graph of a symmetric relation either any two different vertices are non-adjacent or there exists two-sided arrow between them:



One-sided arrows are forbidden in the graph of symmetric relation.

Relation R is called ***anti-symmetric*** if xRy and yRx imply $x = y$ for all $x, y \in A$ $((x, y) \in R \;\&\; (y, x) \in R \rightarrow x = y)$. In other words if $(x, y) \in R$ and $x \neq y$ then $(y, x) \notin R$.

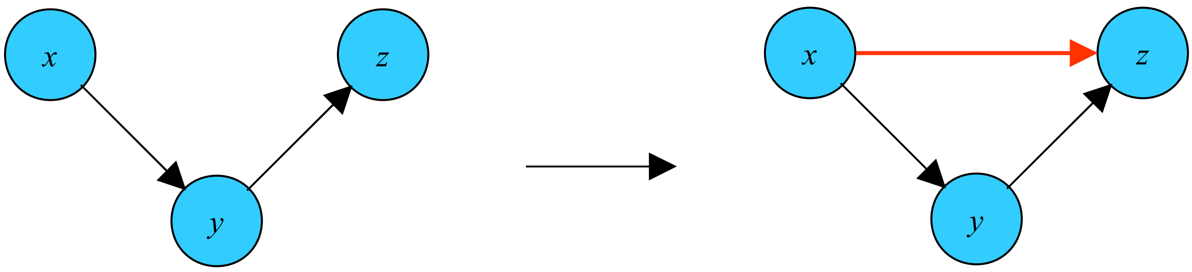
In a matrix of anti-symmetric relation any two elements symmetric with respect to main diagonal must contain zero. In the graph of anti-symmetric relation either any two different vertices are non-adjacent or there exists one-sided arrow between them:



Two-sided arrows are forbidden in the graph of anti-symmetric relation.

Relation R is called ***transitive*** if xRy and yRz imply xRz for all $x, y, z \in A$ $((x, y) \in R \;\&\; (y, z) \in R \rightarrow (x, z) \in R)$.

In the graph of transitive relation if there are two arcs (x, y) and (y, z) then there must be the third arc (x, z):



Example 1.

Consider the following relation R_1 on the set $\mathbb{Z}$ of all integers: xR_1y if and only if $(x - y)$ is even $(x, y \in \mathbb{Z})$.

The relation R_1 is reflexive because $x - x = 0$ is even for any $x \in \mathbb{Z}$

It's symmetric because if $(x - y)$ is even then $(y - x)$ is even, too, for all $x, y \in \mathbb{Z}$.

It's not anti-symmetric because, for instance, $(5 - 3)$ is even and $(3 - 5)$ is even, but $5 \neq 3$.

The relation R_1 is transitive because if $(x - y)$ and $(y - z)$ are even then $x - z = (x - y) + (y - z)$ is also even, for all $x, y, z \in \mathbb{Z}$.

Example 2.

Let R_2 be *divisibility relation* (see the definition above) on the set $\mathbb{Z}$ of all integers: xR_2y if and only if $x|y$ (x divides y), i.e. there exists $k \in \mathbb{Z}$ such that $y = k\,x$ $(x, y \in \mathbb{Z})$.

The relation R_2 is reflexive because $x|x$ $(x = 1 \cdot x)$ for any $x \in \mathbb{Z}$.

It's not symmetric because, for instance, 3 divides 6 but 6 doesn't divide 3.

It's not anti-symmetric because, for instance, 5 divides -5 $(-5 = (-1) \cdot 5)$ and -5 divides 5 $(5 = (-1) \cdot (-5))$ but $5 \neq -5$.

The relation R_2 is transitive because if $x|y$ and $y|z$ then $x|z$ for all $x, y, z \in \mathbb{Z}$: if $y = k\,x$ and $z = m\,y$ for some $k, m \in \mathbb{Z}$ then $z = (mk)\,x$.

Example 3.

Let R_3 be the same divisibility relation as in the previous example, but now let it be given on the set $\mathbb{N}$ of all naturals: xR_3y if and only if $x|y$ (x divides y), i.e. there exists $k \in \mathbb{N}$ such that $y = k\,x$ $(x, y \in \mathbb{N})$.

It's very easy to verify that the relation R_3 given on the set $\mathbb{N}$ is reflexive, transitive and is not symmetric (this verification is absolutely the same as for the divisibility relation on the set $\mathbb{Z}$).

But in comparison with the previous example the relation R_3 of divisibility on the set $\mathbb{N}$ is anti-symmetric: for all $x, y \in \mathbb{N}$ we have

$$xR_3y \text{ and } yR_3x \Leftrightarrow x|y \text{ and } x|y \Leftrightarrow y = k\,x \text{ and } x = m\,y \text{ for some } k, m \in \mathbb{N} \Leftrightarrow$$
$$\Leftrightarrow x = (mk)\,x,\ k, m \in \mathbb{N} \Leftrightarrow mk = 1,\ k, m \in \mathbb{N} \Leftrightarrow m = k = 1 \Leftrightarrow x = y.$$

§4. Equivalence relations and partitions.

A relation R given on a set A is called ***equivalence relation*** if it is reflexive, symmetric and transitive.

For example, the relation R_1 from the previous paragraph is equivalence relation on the set $\mathbb{Z}$ of all integers while both divisibility relations R_2 and R_3 given on the sets $\mathbb{Z}$ and $\mathbb{N}$, correspondently, are not equivalences due to they are not symmetric.

Notion of equivalence relation is closely linked with notion of partition.

Let A be a set. A family P of subsets of A is called a ***partition*** of A if:

1) all the subsets in P are pair-wise disjoint;
2) union of all the subsets from P is equal to A.

In other words, a family P is partition of a set A if every element of A belongs to precisely one set from P.

Elements of P (they are subsets of A) are called *parts* of the partition.

Example 1.

Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$P_1 = \{0, 3, 6, 9\}$, $P_2 = \{1, 2, 4\}$, $P_3 = \{5\}$, $P_4 = \{7, 8\}$.

It's easy to verify that $P_i \cap P_j = \varnothing$ if $i \neq j$ and $P_1 \cup P_2 \cup P_3 \cup P_4 = A$. So, the family $P = \{P_1, P_2, P_3, P_4\}$ is partition of the set A.

Example 2.

Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$P'_1 = \{0, 3, 6, 9\}$, $P'_2 = \{1, 2, 4\}$, $P'_3 = \{5, 9\}$, $P'_4 = \{7, 8\}$.

We see that $P'_1 \cup P'_2 \cup P'_3 \cup P'_4 = A$. But the family $P = \{P'_1, P'_2, P'_3, P'_4\}$ is not a partition of the set A because $P'_1 \cap P'_3 = \{5\} \neq \varnothing$, so element 5 belongs to more than two given subsets of A.

Example 3.

Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$P''_1 = \{0, 3, 6\}$, $P''_2 = \{1, 2, 4\}$, $P''_3 = \{5\}$, $P''_4 = \{7, 8\}$.

We see that $P_i \cap P_j = \varnothing$ if $i \neq j$, but $P_1 \cup P_2 \cup P_3 \cup P_4 = A - \{9\}$, i.e. element 9 doesn't belong to any given subset of A. So, the family $P = \{P''_1, P''_2, P''_3, P''_4\}$ is not a partition of the set A.

Let A be a set and P be partition of A. We can define the following relation R on A: let element x be in the relation R to element y (xRy) if and only if x and y belong to the same part of the partition P. Obviously, R is equivalence relation. The following theorem states that every equivalence relation is constructed in a similar way.

Factorization theorem. Let R be an equivalence relation given on a set A. Then there exists such a partition P of A that xRy holds if and only if x and y belong to the same part of P, for all $x, y \in A$.

To find parts of partition P corresponding to the equivalence relation R it's sufficient to construct the set $P(x) = \{y \in A\colon\ xRy\}$ for any element $x \in A$ and verify that for any two different elements x_1 and x_2 from A either $P(x_1) = P(x_2)$ or $P(x_1) \cap P(x_2) = \varnothing$.

So, if equivalence relation R is defined on a set A then according to the factorization theorem A can be partitioned into several parts in such a way that any two elements from the same part are in R and any two elements from different parts are not in R.

These parts are called ***equivalence classes***.

The set of all equivalence classes is called ***factor set*** of the set A by relation R and is denoted by A/R.

Example 1.

Let R_1 be the following relation on the set $\mathbb{Z}$ of all integers: xR_1y if and only if $(x - y)$ is even $(x, y \in \mathbb{Z})$.

It was proved above that the relation R_1 is equivalence. It's easy to verify that

$P(0) = P(2) = P(-2) = P(4) = P(-4) = \ldots = P_0$ is the set of all even numbers, while

$P(1) = P(-1) = P(3) = P(-3) = \ldots = P_1$ is the set of all odd numbers.

So, P_0 and P_1 are equivalence classes and factor set is the set $\mathbb{Z}/R_1 = \{P_0, P_1\}$.

Example 2.

Suppose $n \in \mathbb{N}$ ($n \geqslant 2$). We define the following relation R_2 on $\mathbb{Z}$: let xR_2y if and only if n divides $(x-y)$, i.e. there exists such integer k that $x - y = k\,n$.

It's not difficult to see that R_2 is generalization of relation R_1. So, it's equivalence relation for any n (the proof is analogous to the case $n = 2$). If xR_2y then one says "x is congruent to y modulo n" and writes $x \equiv y \pmod{n}$.

Two integers are congruent modulo n if and only if they have the same remainder when being divided by n. Hence there are precisely n equivalence classes for the relation R_2 of congruence modulo n:

$P_0 = \{\dots, -3n, -2n, -n, 0, n, 2n, 3n, \dots\}$,

$P_1 = \{\dots, -3n + 1, -2n + 1, -n + 1, 1, n + 1, 2n + 1, 3n + 1, \dots\}$,

$P_2 = \{\dots, -3n + 2, -2n + 2, -n + 2, 2, n + 2, 2n + 2, 3n + 2, \dots\}$,

…

$P_{n-1} = \{\dots, -2n - 1, -n - 1, -1, n - 1, 2n - 1, 3n - 1, \dots\}$.

These equivalence classes are called *residue classes modulo* n.

So, factor set is the set $\mathbb{Z}/R_2 = \{P_0, P_1, P_2, \dots, P_{n-1}\}$ of residue classes modulo n.

§5. Order relations. Ordered sets. Linear and partial orders. Hasse diagram.

A relation R given on a set A is called ***order relation*** if it is reflexive, anti-symmetric and transitive.

If R is order relation and xRy then one says "x precedes y" or "x is less than y".

A set A together with order relation R defined on it is called ***ordered set***. Accurately, ordered set is the pair (A, R).

Example 1.

Let's consider three relations described in §3. The relations R_1 and R_2 are not order relations because they are not anti-symmetric. But the divisibility relation R_3 given on the set $\mathbb{N}$ of all naturals is order. So, the divisibility relation R_3 is order and the pair $(\mathbb{N}, R_3) = (\mathbb{N}, \mid)$ is ordered set.

Example 2.

Let two elements $x, y \in \mathbb{Z}$ be in relation if and only if $x \leqslant y$. It's clear that $x \leqslant x$ (reflexivity), $x \leqslant y$ and $y \leqslant x$ imply $x = y$ (anti-symmetry), $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$ (transitivity). So, the comparability relation $\leqslant$ is order and the pair $(\mathbb{Z}, \leqslant)$ is ordered set.

Example 3.

Let A be a set and 2^A be its power set (the set of all its subsets). Further, let two sets $X, Y \in 2^A$ be in relation if and only if $X \subseteq Y$. It's clear that $X \subseteq X$ (reflexivity), $X \subseteq Y$ and $Y \subseteq X$ imply $X = Y$ (anti-symmetry), $X \subseteq Y$ and $Y \subseteq Z$ imply $X \subseteq Z$ (transitivity). So, the inclusion relation $\subseteq$ is order and the pair $(2^A, \subseteq)$ is ordered set.

There is important distinction between ordered sets $(\mathbb{Z}, \leqslant)$ and $(2^A, \subseteq)$. Namely, for any $x, y \in \mathbb{Z}$ at least one of the inequalities $x \leqslant y$ or $y \leqslant x$ is valid. But for $X, Y \in 2^A$ it can happen that $X \not\subseteq Y$ and $Y \not\subseteq X$. For instance, $A = \{a, b, c\}$, $X = \{a, b\}$, $Y = \{a, c\}$.

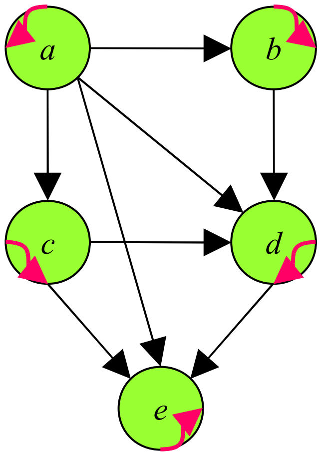
Let (A, R) be ordered set and $x, y \in A$. Elements x and y are called ***comparable*** if either xRy or yRx else they are called ***incomparable***.

In the ordered set $(\mathbb{Z}, \leqslant)$ any two elements are comparable.
In the ordered set $(2^A, \subseteq)$ there exist incomparable elements.
The ordered set $(\mathbb{N}, |\,)$ also has incomparable elements (for instance, 3 and 5).

Order relation R given on a set A is called ***linear order*** if any two elements from A are comparable, else it is called ***partial order***.

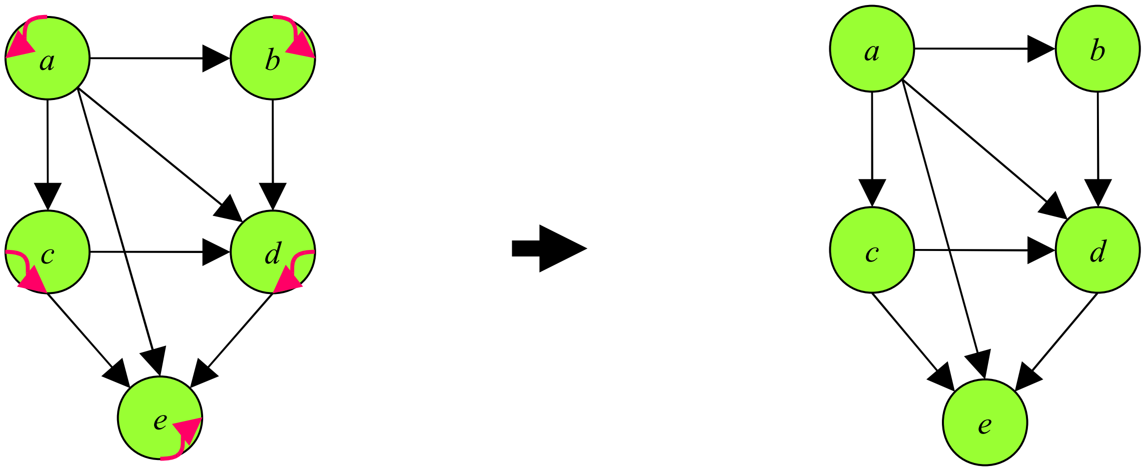
It follows from the definition that $(\mathbb{Z}, \leqslant)$ is linearly ordered set, but $(\mathbb{N}, |\,)$ and $(2^A, \subseteq)$ are partially ordered sets.

The graph of any order relation can be drawn in a simplified form. Let's consider the following example:

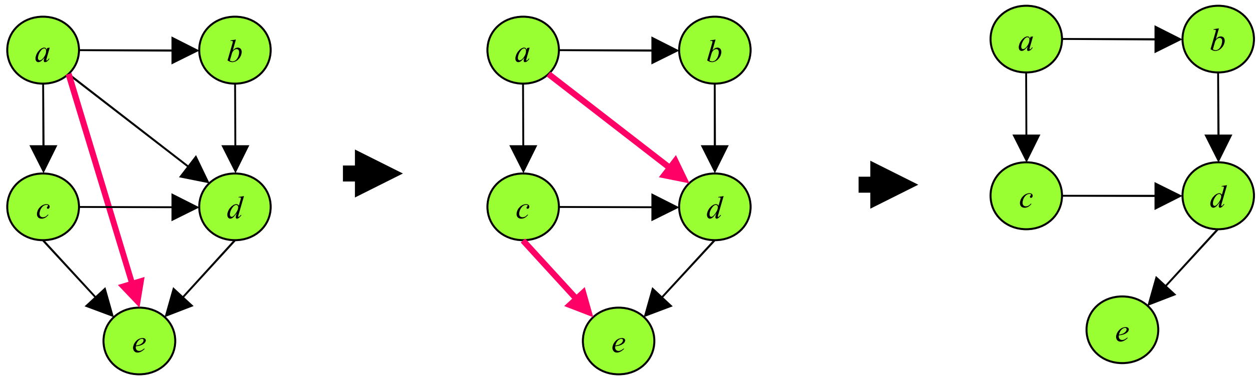


It's not difficult to verify that the given graph represents some order relation on the set $A = \{a, b, c, d, e\}$. Let's apply to the graph the following procedure.

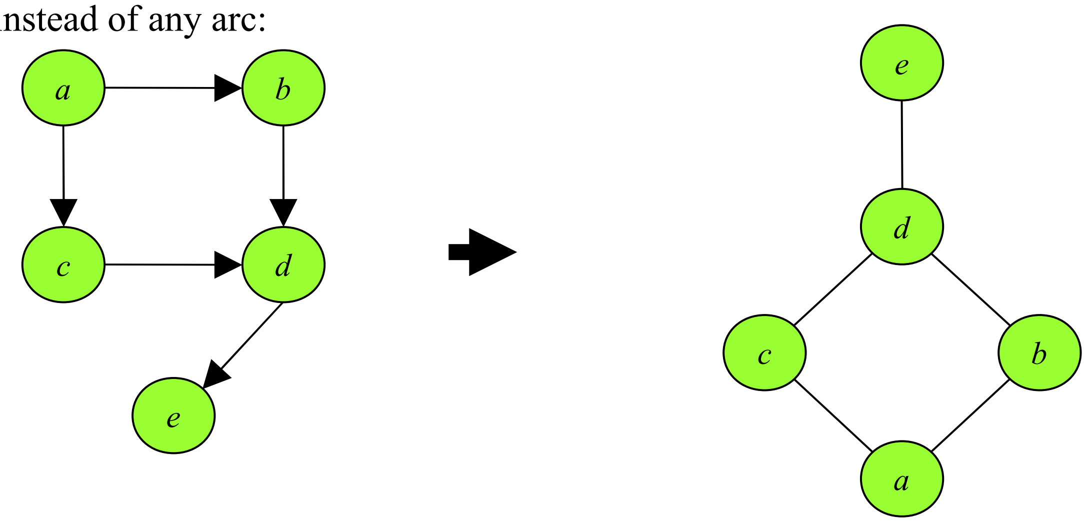
1. Due to reflexivity every vertex has a loop. So, we can omit all these loops:



2. If there are arrows from x to y and from y to z then due to transitivity there must be also the arrow from x to z. So, we can omit all the arrows which existence is a consequence of transitivity:



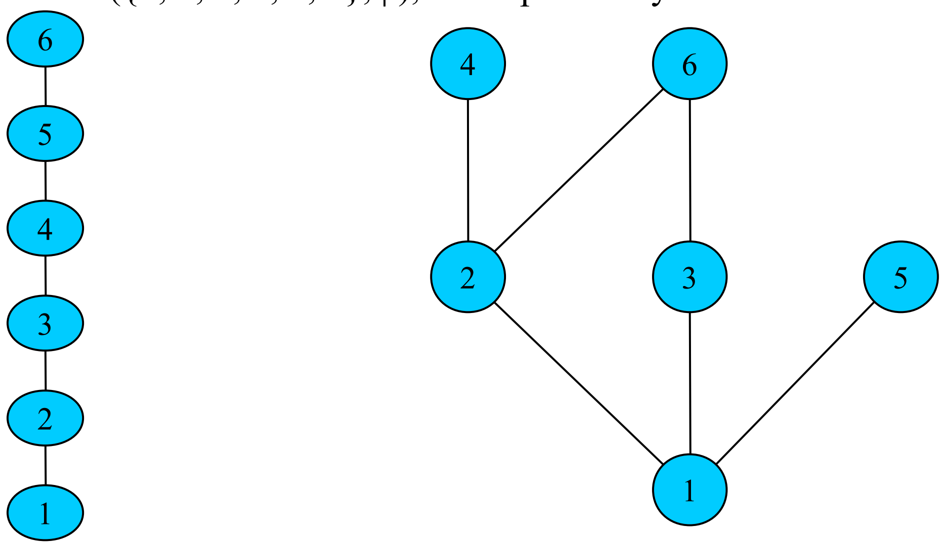
3. We can put “less” element below “greater” one and then we can draw undirected edge instead of any arc:



The obtained undirected graph is called ***Hasse diagram*** of order relation.

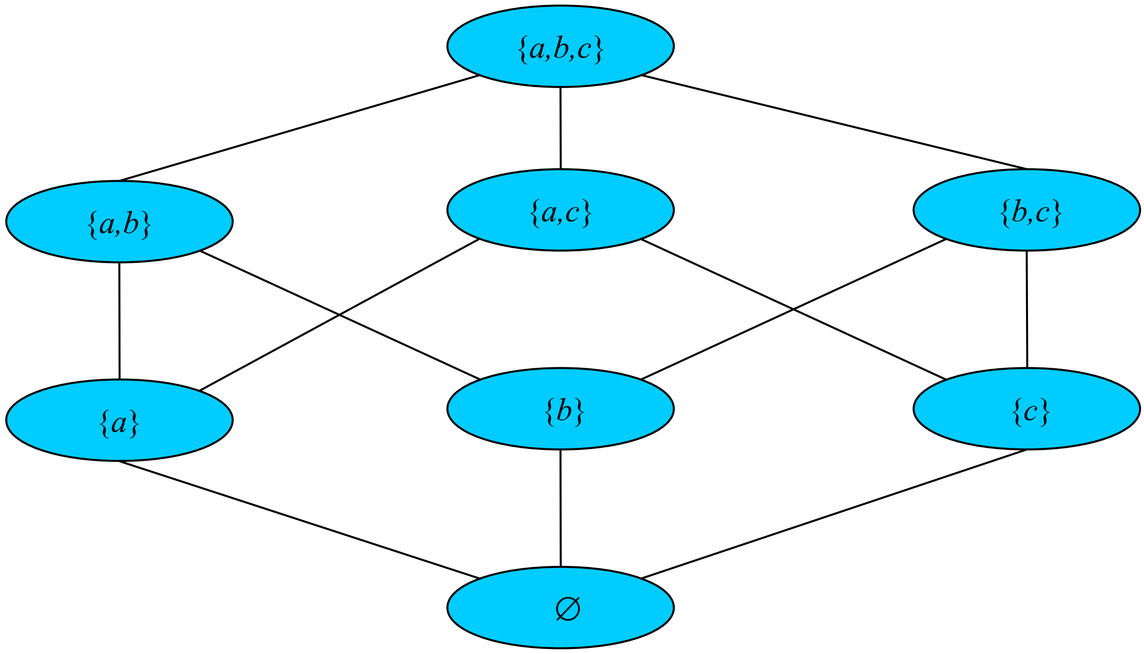
Example 4.

These are Hasse diagrams of linearly ordered set $(\{1, 2, 3, 4, 5, 6\}, \leqslant)$ and partially ordered set $(\{1, 2, 3, 4, 5, 6\}, |)$, correspondently.



Example 5.

That is Hasse diagram of partially ordered set $(2^{\{a, b, c\}}, \subseteq)$.



Element $x \in A$ is called ***maximal*** element of an ordered set (A, R) if there is no element y such that $y \neq x$ and xRy.

In other words, there are no elements “greater” than x under the order R.

Analogously, element $x \in A$ is called ***minimal*** element of an ordered set (A, R) if there is no element y such that $y \neq x$ and yRx.

It means there are no elements “less” than x under the order R.

For instance, it obvious that the ordered set $(\mathbb{Z}, \leqslant)$ doesn't contain maximal and minimal elements, but the set $(\mathbb{N}, \leqslant)$ contains one minimal element (it's 1) and no maximal element.

Every finite ordered set is proved to contain maximal and minimal elements.

Theorem. If (A, R) is finite ordered set and $x \in A$ then there exists maximal element $y \in A$ such that xRy, and there exists minimal element $z \in A$ such that zRx.

Hasse diagram of finite ordered set helps to find easily all its maximal and minimal elements.

Example 6.

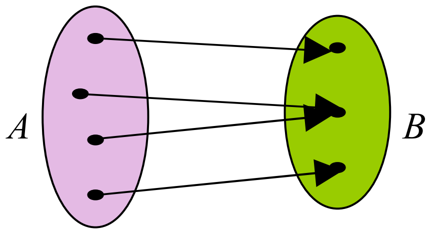
Linearly ordered set $(\{1, 2, 3, 4, 5, 6\}, \leqslant)$ from example 4 has one minimal element (it's 1) and one maximal element (it's 6). Obviously, every finite *linear* order has one minimal and one maximal element.

Partially ordered set $(\{1, 2, 3, 4, 5, 6\}, \mid)$ from the same example has one minimal element (it's 1) and three maximal element (4, 5 and 6).

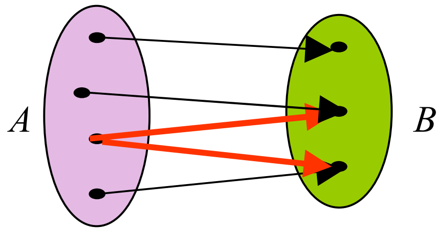
Partially ordered set $(2^{\{a, b, c\}}, \subseteq)$ from example 5 has one minimal element (it's $\varnothing$) and one maximal element (it's the set $\{a, b, c\}$).

§6. Functional relations and functions.

Relation R between sets A and B is called ***functional relation*** if for any $x \in A$ there exists unique $y \in B$ such that xRy.



functional relation



not functional relation

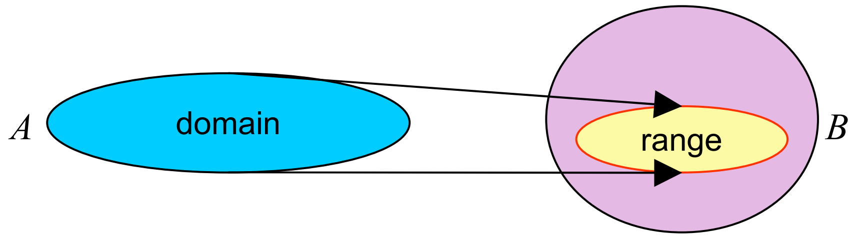
Every functional relation R between A and B is linked with a function $y = f(x)$. Element x is called an argument of the function, it takes values from A. Element y is the function value, it is the element from B for which xRy holds.

One says that f is ***function*** (***mapping***, ***correspondence***, ***transformation***) from A into B and writes $f: A \to B$.

The set A is called ***domain*** of the function.

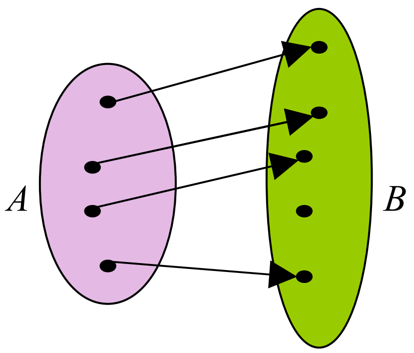
If $X \subseteq A$ then $f(X)$ denotes the set of all elements $y \in B$ for which there exists element $x \in X$ such that $y = f(x)$.

The set $f(A)$ is called ***range*** (***value set***) of the function.

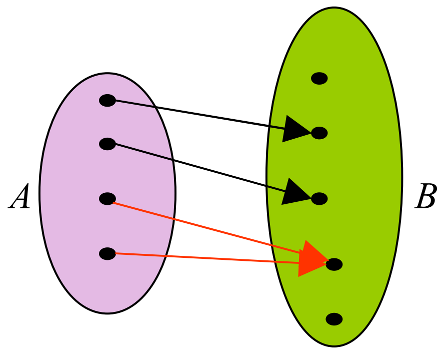


Function $f: A \to B$ is called ***injective function*** (***injection***) if for any $x_1, x_2 \in A$ $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

If A and B are finite sets and there is injection from A to B then $|A| \leqslant |B|$.



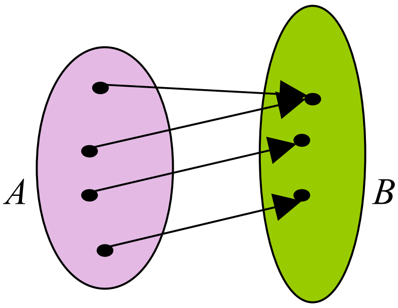
injection



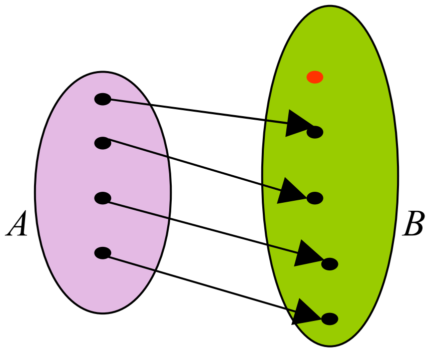
not an injection

Function $f: A \to B$ is called ***surjective function*** (***surjection***) if its range is B, i.e. for any $y \in B$ there exists (not necessarily unique) $x \in A$ such that $y = f(x)$. In the case we say that f is the function from A *onto* B.

If A and B are finite sets and there is surjection from A onto B then $|A| \geqslant |B|$.



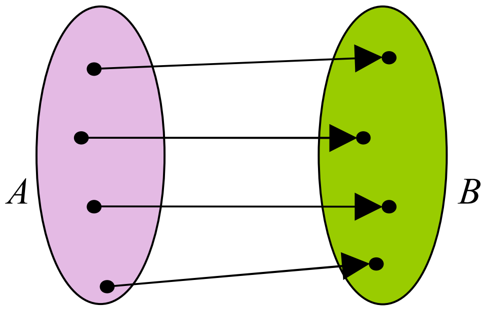
surjection



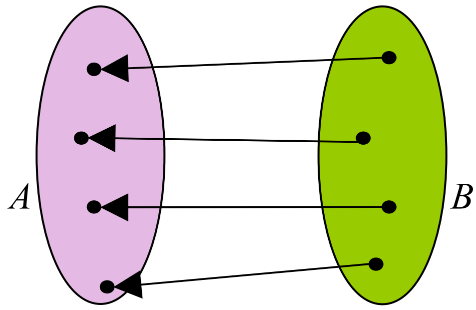
not a surjection

Function $f: A \to B$ is ***bijective function*** (***bijection***) if it is injective and surjective. Bijection is also called ***one-to-one correspondence***.

If $f: A \to B$ is bijection then there exists ***inverse function*** $f^{-1}: B \to A$ such that $f^{-1}(f(x)) = x$ for all $x \in A$ (f^{-1} is denotation for inverse function).



bijection f



inverse function f^{-1} to bijection f

Example.

Let $A = \{-3, -2, 2, 3\}$, $B = \{4, 9\}$, $C = \{-27, -8, 0, 8, 27\}$, $D = \{-27, -8, 8, 27\}$ be sets and let $R_1 \subseteq A \times B$, $R_2 \subseteq A \times B$, $R_3 \subseteq A \times C$, $R_4 \subseteq A \times D$ be binary relations between the set A and one of the sets B, C, D, correspondently, given as the sets of pairs:

$R_1 = \{(-3, 4), (-2, 9), (2, 4), (3, 4), (3, 9)\}$,
$R_2 = \{(-3, 9), (-2, 4), (2, 4), (3, 9)\}$,
$R_3 = R_4 = \{(-3, -27), (-2, -8), (2, 8), (3, 27)\}$.

The relation R_1 is not functional because for the element $3 \in A$ there are two elements $4, 9 \in B$ such that $3R_1 4$ and $3R_1 9$.

The relations R_2, R_3 and R_4 are functional. R_2 corresponds to the function $f(x) = x^2$, R_3 and R_4 correspond to the function $g(x) = x^3$, $x \in A$. The set A is domain of the functions f and g. The set B is range of f, the set D is range of g (note that $D \subset C$).

The function f is surjection. But it is not injection because $f(-3) = f(3) = 9$, $f(-2) = f(2) = 4$ (although $-3 \neq 3$, $-2 \neq 2$).

The function g is injection. It's clear that g is mapping from A onto D. So, if we consider the function as mapping g: $A \to C$ corresponding to the relation R_3 then g is not surjection. But if we consider the function as mapping g: $A \to D$

corresponding to the relation R_4 then g is surjection. It follows that in the second case g is bijection and we can construct its inverse function $g(y) = \square\,\overline{y}$, $y \in D$, taking values from the set A.

If A and B are finite sets and there exists bijection from A to B then $|A| = |B|$. This statement is known as "***equality rule***". It is an important principle of counting. If we know the cardinality of a set A and if there is a way to establish bijection between A and another set B then we know the cardinality of the set B, too.

For instance, if $h(A)$ is the function associating to the characteristic vector of each subset A of a finite universe U then $h(A)$ is bijection from the power set 2^U of the set U onto the set $\{0, 1\}^n$ where $n = |U|$:

$$h\colon\ 2^U \to \{0, 1\}^n.$$

Hence, by the equality rule, the subset number of a set U having cardinality n is equal to the number of sequences having the length n consisting of zeroes and ones (and is equal to 2^n).

The concepts of injection and bijection allow us to compare infinite sets. If there is *injection* from A to B then it is naturally to think that the cardinality of A is not greater than the cardinality of B. If there is *bijection* between A and B then the cardinalities of A and B are equal.

Infinite set A is called ***countable*** if there is a bijection between A and the set $\mathbb{N}$ of all naturals, else it is ***uncountable***.

If A is countable and a function $f: \mathbb{N} \to A$ is bijection then we can arrange all elements from A in an infinite sequence $f(1)$, $f(2)$, $f(3)$, … Conversely, if there exists such a sequence that contains all elements of A then A is countable.

Theorem. The sets $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable. The set $\mathbb{R}$ of all reals is uncountable.

Chapter 3. Elements of combinatorics.

§1. Basic principles of counting: the sum rule and the product rule.

We already know one of counting methods – the equality rule. There are two other very useful combinatorial tools.

The sum rule. If A and B are disjoint finite sets ($A \cap B = \varnothing$) then
$$|A \cup B| = |A| + |B|.$$

The product rule. If A and B are arbitrary finite sets then $|A \times B| = |A| \cdot |B|$ ($A \times B$ is Cartesian product of the sets A and B).

Example.

Suppose there are m books written by first author and n books written by second author. How many ways are there to take a book of any author? What is the number of ways to take one book of the first author and one book of the second author?

Solution.

Let A be the set of books written by the first author and B be the set of books written by the second author. Obviously, $|A| = m$, $|B| = n$ and $A \cap B = \varnothing$.

To take a book of any author means to eject one element from the set $A \cup B$. According to the sum rule it can be done by $|A \cup B| = |A| + |B| = m + n$ ways.

To take one book of the first author and one book of the second author means to eject one element (to eject a pair) from the set $A \times B$. According to the product rule it can be done by $|A \times B| = |A| \cdot |B| = mn$ ways.

Both rules can be applied to any number of sets. So, we get generalized sum and product rules.

Generalized sum rule. If $A_1, A_2, \ldots, A_k$ are pairwise disjoint finite sets (i.e. $A_i \neq A_j$ if $i \neq j$) then $|A_1 \cup A_2 \cup \ \ldots \cup A_k| = |A_1| + |A_2| + \ \ldots + |A_k|$.

Generalized sum rule. If $A_1, A_2, \ldots, A_k$ are arbitrary finite sets then
$$|A_1 \times A_2 \times \ \ldots \times A_k| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_k|\ .$$

For instance, as a consequence of the general product rule we get the following

Theorem. The number of sequences having the length n and consisting of elements of a set A having the cardinality k is equal to k^n.

In terms of sets it means that if $|A| = k$ then $|A^n| = k^n$.

§2. Permutations, permutation number.

Permutation of elements of a set A is a sequence in which each element from A occurs precisely once. In other words, a permutation is some arrangement of all elements of the set A. Various permutations of A differ only by order of elements.

What is the number of different permutations of n-element set?

Without loss of generality we can assume that $A = \{1, 2, \ldots, n\}$.
It is possible to order two elements by the two ways: (1, 2) and (2, 1).
For $n = 3$ there are 6 permutations:

$$(1, 2, 3),\ (1, 3, 2),\ (2, 1, 3),\ (2, 3, 1),\ (3, 1, 2),\ (3, 2, 1).$$

Let's introduce the denotation $n! = 1 \cdot 2 \cdot 3 \quad (n-1) \cdot n$. It's read "$n$ factorial".

Theorem. The permutation number of n-element set is equal to $n!$

***k-element permutation* (*k-permutation*)** of a set A ($|A| = n$, $k \leqslant n$) is a sequence of length k in which each element from A occurs at most once. Particularly, n-permutation of a set having n elements is ordinary permutation.

For instance, there are 12 2-permutations of the set having 4 elements:

$$(1, 2),\ (2, 1),\ (3, 1),\ (4, 1),\ (1, 3),\ (2, 3),$$
$$(3, 2),\ (4, 2),\ (1, 4),\ (2, 4),\ (3, 4),\ (4, 3).$$

Theorem. The number of all k-permutations of n-element set is equal to

$$P(n, k) = n \cdot (n-1) \cdot (n-2) \ldots (n-k+1) = n!/(n-k)!$$

Example 1.

Suppose, n sportsmen take part in a competition. What is the number of ways to choose from them 3 winners (sportsmen who win the 1-st place, the second place and the third place) if for every chosen sportsman it's necessary to point the place that he (or she) wins?

Solution: the number of ways satisfying the described condition to choose 3 winners equals $P(n, 3) = n \cdot (n-1) \cdot (n-2) = n!/(n-3)!$

§3. Combinations, combination number, properties of combination numbers.

Let A be arbitrary set of cardinality n, $0 \leqslant k \leqslant n$. ***k-combination*** of A is any k-subset of A.

Concepts of permutation and combination are essentially different because any k-permutation is ordered collection, it's a sequence, but any k-combination is unordered collection, it's some set (subset of A). If we have some k-combination then we can obtain $k!$ distinct k-permutations from it. For instance, 3-combination $\{1, 2, 3\}$ induces the following 6 3-permutations:

$$(1, 2, 3),\ (1, 3, 2),\ (2, 1, 3),\ (2, 3, 1),\ (3, 1, 2),\ (3, 2, 1).$$

So, to obtain the set of all k-permutations we need to take any k-combination and to permutate its elements. The latter can be done by $k!$ ways. Consequently,

$$P(n, k) = k! \cdot C(n, k),$$

where $C(n, k)$ is denotation for combination numbers (it's the number of all k-element subsets of n-element set). Thus, we proved the following

Theorem. The number of all k-combinations of n-element set is equal to

$$C(n, k) = P(n, k)/k! = n\cdot(n-1)\cdot(n-2)\ \dots\ (n-k+1)/k! = n!/(k!\cdot(n-k)!)$$

The combination numbers $C(n, k)$ are called ***binomial coefficients***.

Example 2.

Let's consider again n sportsmen taking part in a competition. What is the number of ways to choose from them 3 winners provided it doesn't matter what the place of any chosen sportsman is?

Solution: the number of ways to choose 3 winners without pointing their places is the number of all 3-subsets of n-element set, it equals

$$C(n, 3) = n!/(3!\cdot(n-3)!) = n\cdot(n-1)\cdot(n-2)/6.$$

Example 3.

A student wants to select 3 days from 5 (Monday, Tuesday, Wednesday, Thursday, Friday) for working in a library. What is the number of ways to do this?

Solution: the problem is to select 3-subset from a set of size 5. There are

$$C(5, 3) = 5!/(3!\cdot 2)!) = 4\cdot 5)/2 = 10$$

such subsets. We can enumerate these 3-subsets:

{Mon, Tue, Wed}, {Mon, Tue, Thu}, {Mon, Tue, Fri}, {Mon, Wed, Thu},
{Mon, Wed, Fri}, {Mon, Thu, Fri}, {Tue, Wed, Thu},
{Tue, Wed, Fri}, {Tue, Thu, Fri}, {Wed, Thu, Fri}.

The combination numbers have several important properties. Some of them are described below.

Theorem (*properties of combination numbers*).

1°. $C(n, 0) = C(n, n) = 1$.

2°. $C(n, 1) = C(n, n-1) = n$.

3°. $C(n, k) = C(n, n-k)$
(it's *symmetry of binomial coefficients*).

4°. $C(n, k) = (n/k)\cdot C(n-1, k-1)$.

5°. $C(n, k) = C(n-1, k) + C(n-1, k-1)$
(this equality is known as *Pascal triangle*).

§4. Binomial theorem and corollaries.

There are well-known formulae:

$$(a+b)^2 = a^2 + 2ab + b^2,\quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

It's also not difficult to verify the validity of the formulae

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5a^2b^3 + b^5,$$

and so on. We can see that the coefficients in the right sides of these equalities are the combination numbers.

The formulae written above can be generalized for any value of power n. As a result we get the statement of very important theorem in combinatorics.

Binomial Theorem (Newtonian Binomial).

$$(a + b)^n = \sum_{k=0}^{n} C(n, k)\cdot a^k\, b^{n-k} \quad \text{for any natural } n \text{ and real } a, b.$$

Several identities concerning the binomial coefficients can be derived from the Binomial Theorem. There are two of them.

Corollary 1.

Let $a = b = 1$. Then we obtain: $\sum_{k=0}^{n} C(n, k) = 2^n$.

This equality has very simple interpretation: due to $C(n, k)$ is the number of all k-element subsets of n-element set it follows that left side calculates the number of all subsets of the set. But according to the theorem on the cardinality of power set this number equals 2^n.

Corollary 2.

Let $a = -1$, $b = 1$. Then we have: $\sum_{k=0}^{n} (-1)^k \cdot C(n, k) = 0$.

If we rewrite this equality in the form $\sum_{k \text{ is even}} C(n, k) = \sum_{k \text{ is odd}} C(n, k)$ then we see it expresses the fact that the number of subsets having even cardinality is equal to the number of subsets with odd cardinality.

§5. Partitions with given specification.

$C(n,k)$ is the number of subsets having the size k of a set having the size n. We can also consider $C(n,k)$ as the number of partitions of a set with n elements into two parts, the first consisting of k elements and the second consisting of $(n - k)$ elements. Now we shall consider partitions into s parts.

Let A be a set of cardinality n and numbers $k_1, k_2, \ldots, k_s \in \mathbb{N}_0$ satisfy the condition $k_1 + k_2 + \ldots + k_s = n$. Partition $A = P_1 \cup P_2 \cup \ldots \cup P_s$ is called ***partition with the specification*** $(k_1, k_2, \ldots, k_s)$ if $|P_1| = k_1$, $|P_2| = k_2, \ldots, |P_s| = k_s$.

Problem. What is the number of all partitions with the given specification?

The following statement is the answer to the question/

Theorem. The number of all partitions of n-element set with the specification $(k_1, k_2, \ldots, k_s)$ equals

$$C(n; k_1, k_2, \ldots, k_s) = n! / (k_1!\, k_2! \ldots k_s!).$$

The number $C(n; k_1, k_2, \ldots, k_s)$ is called ***polynomial coefficient***.

Example.

There are 10 teachers, each of them delivers lectures in algebra, geometry and physics. It's necessary to form 3 committees from them to check student tests in these subjects. The committee in algebra must have 5 teachers, the committee in geometry – 3 teachers, and the committee in physics – 2 teachers. Every teacher must be represented in only one committee. What is the number of ways to form such 3 committees?

Solution. We have 10-element set with the specification (5, 3, 2), i.e. we should partition it by 3 parts with cardinalities 5, 3, 2, correspondently. It can be done by $C(10; 5, 3, 2) = 10!/(5!\ 3!\ 2!)$ ways.

§6. Inclusion-exclusion method.

Let's revise the sum rule: for any disjoint finite sets A and B ($A \cap B = \varnothing$) the equality $|A \cup B| = |A| + |B|$ holds.

If $A \cap B \neq \varnothing$ then the equality fails but we can easily correct it. In fact, elements of $A \cap B$ are counted twice in the sum $|A| + |B|$. Hence to obtain true equality it is sufficient to subtract $|A \cap B|$: for any two sets A and B we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Analogously, for any three sets A, B and C we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Now we shall formulate the inclusion-exclusion principle in general form. Let $A_1, A_2, \ldots, A_n$ be finite sets. Let's call an intersection of k these sets *k-intersection.* It is *even* or *odd* intersection according to k is even or odd.

Theorem. $|A_1 \cup A_2 \cup \ldots \cup A_n| = S_{\text{odd}} - S_{\text{even}}$

where S_{odd} is the sum of cardinalities for all odd intersections and S_{even} is the sum of cardinalities for all even intersections.

Example.

There are 32 students in a group, each of them attends a special course in chemistry or in biology or in astronomy. 20 students attend the course in chemistry, 15 – in biology. 11 students attend courses in chemistry and biology, 8 – in chemistry and astronomy, 5 – in biology and astronomy. 3 students attend all the three special courses. How many students do attend the course in astronomy? How many students do attend precisely 2 courses? Precisely 1 course?

Solution.

Let A be the set of students attending the course in astronomy, B – in biology, C – in chemistry. Then we have $|B| = 15$, $|C| = 20$, $|A \cap B| = 5$, $|A \cap C| = 8$, $|B \cap C| = 11$, $|A \cap B \cap C| = 3$, $|A \cup B \cup C| = 32$. So, from the inclusion-exclusion formula for the case of 3 sets (see the formula above) it follows that

$|A| = |A \cup B \cup C| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| =$

$= 32 - 15 - 20 + 5 + 8 + 11 - 3 = 18.$

The set $A \cap B \cap \overline{C}$ consists of students who attend the courses in astronomy, biology, but don't attend chemistry, the set $A \cap \overline{B} \cap C$ – astronomy, chemistry, but not biology, and $\overline{A} \cap B \cap C$ – biology, chemistry, but not astronomy. It follows that the set $(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$ contains those students who attend precisely 2 of the three courses. It's easy to see that

$|A \cap B \cap \overline{C}| = |A \cap B| - |A \cap B \cap C| = 5 - 3 = 2,$

$|A \cap \overline{B} \cap C| = |A \cap C| - |A \cap B \cap C| = 8 - 3 = 5,$

$|\overline{A} \cap B \cap C| = |B \cap C| - |A \cap B \cap C| = 11 - 3 = 8.$

So, there are $2 + 5 + 8 = 15$ students who attend precisely 2 of the three courses.

Analogously, the set $A \cap \overline{B} \cap \overline{C}$ consists of students who attend the course in astronomy, but don't attend biology and chemistry, the set $\overline{A} \cap B \cap \overline{C}$ – biology, but not astronomy and chemistry, and the set $\overline{A} \cap \overline{B} \cap C$ – chemistry, but not astronomy and biology. It follows the set $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$ contains those students who attend precisely 1 of the three courses. So, we have

$|A \cap \overline{B} \cap \overline{C}| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C| = 18 - 5 - 8 + 3 = 8,$

$|\overline{A} \cap B \cap \overline{C}| = |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C| = 15 - 5 - 11 + 3 = 2,$

$|\overline{A} \cap \overline{B} \cap C| = |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 20 - 8 - 11 + 3 = 4.$

So, there are $8 + 2 + 4 = 14$ students who attend precisely 1 of the three courses.

Chapter 4. Recurrence equations.

Let $(x_0, x_1, x_2, ...)$ be infinite sequence of numbers where several initial elements $x_0, x_1, ..., x_k$ are given and every element is defined by previous elements according to some rule. This rule is called recurrence (***recurrence equation***).

We shall consider ***linear recurrences with constant coefficients***, i.e. equations where the rule is described by some linear expression.

Order of linear recurrence is the number of previous elements that are required to determine its every current element.

§1. First order linear recurrences.

General ***linear first order recurrence*** is the equation

$$x_n = ax_{n-1} + b,$$

where a and b are given (real) constants, $n \in \mathbb{N}$.

If the initial element x_0 is also given then we can compute sequentially all the other elements of the recurrence:

$$x_1 = ax_0 + b,\ x_2 = ax_1 + b = a(ax_0 + b) + b, \text{ and so on.}$$

The fact is that any element x_n, $n \in \mathbb{N}$, is uniquely defined by a, b and x_0. To write down a general formula for x_n we consider first the following two special cases.

Case 1. Let $a = 1$.

Then we have the recurrence $x_n = x_{n-1} + b$. So, we get

$x_1 = x_0 + b$, $x_2 = x_1 + b = x_0 + 2b$, $x_3 = x_2 + b = x_0 + 3b$, and so on.

Obviously, by induction for any $n \in \mathbb{N}$ we obtain

$$x_n = x_0 + bn.$$

This sequence is *arithmetic progression*.

Case 2. Let $b = 0$.

Then we have the recurrence $x_n = ax_{n-1}$. So, we get

$x_1 = ax_0$, $x_2 = ax_1 = a^2x_0$, $x_3 = ax_2 = a^3x_0$, and so on.

Obviously, by induction for any $n \in \mathbb{N}$ we obtain

$$x_n = x_0 a^n.$$

This sequence is *geometic progression*.

Now let's consider the **general case** $x_n = ax_{n-1} + b$ supposing that $a \neq 1$.

First we'll reduce the equation to a simplified form with a help of the substitution

$$x_n = y_n + s,$$

where y_n is new unknown (it's the n-th member of the sequence $y_0, y_1, y_2, \ldots$) and s is some constant (we'll calculate its value later). After the substitution we obtain

$$y_n + s = a(y_{n-1} + s) + b, \quad \text{or}$$

$$y_n = a y_{n-1} + (as + b - s).$$

Let's select the constant s in such a way to eliminate the expression in parentheses:

$$s = b/(1-a) \quad \text{(note that } s \text{ exists due to } a \neq 1\text{)}.$$

Then in new variables we get the recurrence $y_n = a y_{n-1}$ described in the case 2. According to this case the solution of the latter recurrence is $y_n = y_0 a^n$, $n \in \mathbb{N}$. It remains to return to the original variable x_n. Due to $y_n = x_n - s$ we get

$$x_n - s = (x_0 - s) a^n, \quad n \in \mathbb{N}.$$

Expressing x_n from the last equality and substituting the value s we obtain

$$x_n = (x_0 - b/(1-a)) a^n + b/(1-a), \quad n \in \mathbb{N}.$$

In fact ther is no necessity to memorize this formula. It is rather the solution method. It includes the following three stages:

1. Reduction of equation to a simplest form by the substitution $x_n = y_n + s$ and choice of suitable value for the constant s.
2. Solving of the obtained simplest equation.
3. Returning to the former unknown x_n.

Note that the solution has the form $x_n = c_1 a^n + c_2$, where c_1 and c_2 are some constants. We see that the dependence of x_n on n is expressed by exponential function.

Example. Solve first order linear recurrence

$$x_n = 4x_{n-1} + 7 \text{ under initial condition } x_0 = 5.$$

First we substitute the unknown $x_n = y_n + s$ and get

$$y_n + s = 4(y_{n-1} + s) + 7. \text{ It implies } y_n = 4y_{n-1} + (3s + 7).$$

So, suitable value for the constant is $s = -7/3$. Then we solve the simplest linear first order recurrence $y_n = 4y_{n-1}$ and obtain $y_n = y_0 4^n$. At last, we return to the former unknown: $x_n - s = (x_0 - s) 4^n$. So, we have

$$x_n = (x_0 - s) 4^n + s = (5 + 7/3) 4^n - 7/3 = (22/3) 4^n - 7/3, \quad n \in \mathbb{N}.$$

Second order linear recurrences.

General ***linear second order recurrence*** is an equation

$$x_n = ax_{n-1} + bx_{n-2} + c,$$

where a, b and c are given (real) constants, $n \in \mathbb{N}$.

The linear recurrence is called ***homogeneous*** if $c = 0$. Let's consider first of all homogeneous equations.

§2. Second order homogeneous linear recurrences.

Homogeneous linear second order equation has the form

$$x_n = ax_{n-1} + bx_{n-2}.$$

If we know the two initial elements x_0 and x_1 then it's possible to compute sequentially the other elements:

$x_2 = ax_1 + bx_0$, $x_3 = ax_2 + bx_1 = a(ax_1 + bx_0) + bx_1 = (a^2 + b)x_1 + abx_0$, and so on.

Every element x_n, $n \in \mathbb{N}$, is uniquely defined by a, b, x_0, x_1. As in the case of first order linear recurrences we can obtain general formula for x_n.

We'll look for a solution x_n in the exponential form:

$$x_n = \lambda^n, \quad \text{where } \lambda \text{ is some constant}$$

(λ doesn't depend on n, its value will be found later). This idea follows from the solution form for the case of first order recurrences. Substitution of this expression to the initial equation leads to the equality

$$\lambda^n = a\lambda^{n-1} + b\lambda^{n-2}.$$

Reducing it by λ^{n-2} we obtain the so-called ***characteristic equation***

$$\lambda^2 - a\lambda - b = 0.$$

Thus, λ must be a root of characteristic equation. There are two possibilities.

Case 1. The characteristic equation has two distinct roots, λ_1 and λ_2.

Then it can be verified that the solution x_n can be written in the form

$$x_n = c_1(\lambda_1)^n + c_2(\lambda_2)^n,$$

where c_1 and c_2 are some constants that depend on a, b, x_0, x_1.

Case 2. The characteristic equation has two equal roots, $\lambda_1 = \lambda_2$.

It means that the *discriminant* of the characteristic equation is zero: $a^2 + 4b = 0$. Then the solution proves to have the form

$$x_n = (c_1 + c_2 n)(\lambda_1)^n,$$

where constants c_1 and c_2 again can be found from initial conditions x_0, x_1 and from the coefficients a, b.

So we have the following algorithm for solving linear homogeneous second order recurrence equation:

1. Write down the characteristic equation and solve it.

2a. If the characteristic equation has two distinct roots λ_1 and λ_2 then write down *general solution* in the form

$$x_n = c_1(\lambda_1)^n + c_2(\lambda_2)^n.$$

2b. If the characteristic equation has unique root λ_1 then write down *general solution* in the form

$$x_n = (c_1 + c_2 n)(\lambda_1)^n.$$

3. Write down a system of two linear equations for c_1 and c_2 using given initial values x_0, x_1 and solve it (put $n = 0$ and $n = 1$ to the formula expressing general solution).
4. Substitute found values of the constants c_1 and c_2 to the general solution.

Example 1. Solve second order linear homogeneous recurrence

$$x_n = 2x_{n-1} + 3x_{n-2} \quad \text{under initial conditions } x_0 = 1,\ x_1 = 2.$$

Solution: we write the characteristic equation $\lambda^2 - 2\lambda - 3 = 0$, find its roots $\lambda_1 = 3$, $\lambda_2 = -1$ and write down the general solution $x_n = c_1 3^n + c_2(-1)^n$. Then we put $n = 0$ and $n = 1$ to the latter formula and obtain a system for finding the constants c_1 and c_2:

$$\begin{cases} c_1 + c_2 = 1, \\ 3c_1 - c_2 = 2. \end{cases}$$

The solution of this system is $c_1 = 3/4$, $c_2 = 1/4$.

So, the solution of the given second order homogeneous recurrence is

$$x_n = (3/4)\,3^n + (1/4)\,(-1)^n = (3^{n+1} + (-1)^n)/4,\ n \in \mathbb{N}.$$

Example 2. Solve second order linear homogeneous recurrence

$$x_n = 4x_{n-1} - 4x_{n-2} \quad \text{under initial conditions } x_0 = 1,\ x_1 = 4.$$

Solution: we write the characteristic equation $\lambda^2 - 4\lambda + 4 = 0$, it has two equal roots $\lambda_1 = \lambda_2 = 2$. Thus, the general solution has the form $x_n = (c_1 + c_2 n)\,2^n$. Then we put $n = 0$ and $n = 1$ to the latter formula and obtain a system for finding the constants c_1 and c_2:

$$\begin{cases} c_1 = 1, \\ 2c_1 + 2c_2 = 4. \end{cases}$$

The solution of this system is $c_1 = 1$, $c_2 = 1$.

So, the solution of the given second order homogeneous recurrence is

$$x_n = (1 + n)\,2^n,\ n \in \mathbb{N}.$$

§3. Second order inhomogeneous linear recurrences.

Let's consider now inhomogeneous linear second order recurrence

$$x_n = ax_{n-1} + bx_{n-2} + c,$$

where a, b and c are given (real) constants, $c \neq 0$, $n \in \mathbb{N}$.

This equation proves to be reduced to homogeneous equation in the same way as in the case of the first order recurrence. But for the second order recurrences the choice of unknown substitution depends on the value of $a + b$.

Case 1. Suppose $a + b \neq 1$.

We introduce new unknown y_n such that $x_n = y_n + s$ and look for suitable value of the constant s:

$$y_n + s = a(y_{n-1} + s) + b(y_{n-2} + s) + c.$$

This implies

$$y_n - a y_{n-1} - b y_{n-2} = (a + b - 1)s + c.$$

So, we should take

$$s = c/(1 - a - b)$$

(note that denominator is not zero) and get homogeneous second order recurrence

$$y_n = ay_{n-1} + by_{n-2}.$$

After we solve it we should return to the former unknown by means of inverse substitution $y_n = x_n - s$.

Case 2. Now let $a + b = 1$. So, $a = 1 - b$.

We introduce new unknown y_n such that $x_n = y_n + sn$ and look for suitable value of the constant s:

$$y_n + sn = a(y_{n-1} + s(n-1)) + b(y_{n-2} + s(n-2)) + c.$$

This implies

$$y_n - a y_{n-1} - b y_{n-2} = as(n-1) + bs(n-2) - sn + \text{c} =$$

$$= (a + b - 1)sn - (a + 2b)s + c = c - (b + 1)s.$$

So, if we exclude the only special case $a = 2$, $b = -1$ then we should take the constant

$$s = c/(b + 1)$$

(note that denominator is not zero) and get homogeneous second order recurrence

$$y_n = ay_{n-1} + by_{n-2}.$$

After we solve it we should return to the former unknown by means of inverse substitution $y_n = x_n - sn$.

Example 1. Solve second order linear inhomogeneous recurrence

$$x_n = 6x_{n-1} - 9x_{n-2} + 5 \quad \text{under initial conditions } x_0 = 1,\ x_1 = -1.$$

Solution: $a = 6$, $b = -9$, $a + b = -3 \neq 1$.
So, we use the substitution $x_n = y_n + s$ and get

$$y_n + s = 6(y_{n-1} + s) - 9(y_{n-2} + s) + 5.$$

This implies

$$y_n - 6y_{n-1} + 9y_{n-2} = -4s + 5.$$

So, we should take $s = 5/4$ to get second order homogeneous recurrence

$$y_n - 6y_{n-1} + 9y_{n-2} = 0.$$

It's characteristic equation $\lambda^2 - 6\lambda + 9 = 0$ has equal roots $\lambda_1 = \lambda_2 = 3$, consequently, its general solution has the form $y_n = (c_1 + c_2 n)\, 3^n$. Now we return to former unknown by means of inverse substitution $y_n = x_n - 5/4$ and obtain

$$x_n = (c_1 + c_2 n)\, 3^n + 5/4.$$

It remains to find the coefficients c_1 and c_2. We put $n = 0$ and $n = 1$ to the latter formula and obtain a system for their finding:

$$\begin{cases} c_1 \qquad + 5/4 = 1, \\ 3c_1 + 3c_2 + 5/4 = -1. \end{cases}$$ The solution of this system is $c_1 = -1/4$, $c_2 = -1/2$.

So, the solution of the given second order inhomogeneous recurrence is

$$x_n = (5 - (1 + 2n)\, 3^n)/4, \ n \in \mathbb{N}.$$

Example 2. Solve second order linear inhomogeneous recurrence

$$3x_n = 8x_{n-1} - 5x_{n-2} + 4 \quad \text{under initial conditions } x_0 = 1,\ x_1 = -1/3.$$

Solution: $a = 8/3$, $b = -5/3$, $a + b = 1$.
So, we use the substitution $x_n = y_n + sn$ and get

$$3(y_n + sn) = 8\,(y_{n-1} + s\,(n-1)) - 5\,(y_{n-2} + s\,(n-2)) + 4.$$

This implies

$$3y_n - 8\,y_{n-1} + 5\,y_{n-2} = 2s + 4.$$

So, we should take $s = -2$ to get second order homogeneous recurrence

$$3y_n - 8\,y_{n-1} + 5\,y_{n-2} = 0.$$

It's characteristic equation $3\lambda^2 - 8\lambda + 5 = 0$ has different roots $\lambda_1 = 5/3$, $\lambda_2 = 1$, consequently, its general solution has the form $y_n = c_1\,(5/3)^n + c_2$. Now we return to former unknown by means of inverse substitution $y_n = x_n + 2n$ and obtain

$$x_n = c_1\,(5/3)^n + c_2 - 2n.$$

It remains to find the coefficients c_1 and c_2. We put $n = 0$ and $n = 1$ to the latter formula and obtain a system for their finding:

$$\begin{cases} c_1 + c_2 \quad = 1, \\ (5/3)\,c_1 + c_2 - 2 = -1/3. \end{cases}$$ The solution of this system is $c_1 = 1$, $c_2 = 0$.

So, the solution of the given second order inhomogeneous recurrence is

$$x_n = (5/3)^n - 2n, \ n \in \mathbb{N}.$$

Section III. Main topics of practice in Discrete Mathematics.

Chapter 1. Sets and set operations.

Finding characteristic vectors of subsets. Finding union, intersection, difference, symmetric difference of two sets. Finding complement and power set of a set. Solving set equations: types of solutions (chain of inclusion, parameterized form); necessary and sufficient conditions for existence of solution; verification of solution. Finding Cartesian product of sets, Cartesian degree of a set.

Chapter 2. Binary relations.

Constructing matrix and graph of binary relation. Verification the properties of reflexivity, symmetry, anti-symmetry, transitivity of binary relations. Verification of equivalence relations and constructing factor set. Verification of order relations, linear and partial orders, constructing Hasse diagram of ordered set, finding its maximal and minimal elements. Verification of injections, surjections, bijections.

Chapter 3. Elements of combinatorics.

Solving variety of combinatorial problems with a help of permutations and combinations. Application of partitions with given specification for solving combinatorial problems. Problems using inclusion-exclusion method.

Chapter 4. Recurrence equations.

Solving first order linear recurrences. Solving second order linear homogeneous and inhomogeneous recurrences.

Section IV. Examination questions in Discrete Mathematics.

1. Set, its cardinality. Relations of belonginess and inclusion. Operations with sets: union, intersection, difference, complement, symmetric difference. Properties of union and intersection: commutativity, associativity, distributive laws. Generalized commutativity and associativity. Graphical representation (Venn diagram) for union and intersection.
2. Properties of difference and symmetric difference, commutativity and associativity of symmetric difference, De Morgan laws. Graphical representation (Venn diagram) for difference, complement and symmetric difference. Connection between difference and inclusion of sets. Connection between symmetric difference and equality of sets.
3. Concepts of Cartesian product, Cartesian square, Cartesian n-th degree. Family of subsets of a set. Concept of power set, theorem on the element number in the power set with n elements. Representation of subsets of finite sets: characteristic vector of a subset, binary tree.
4. Set equations. Algorithm of their solving. Necessary and sufficient conditions for existence of a solution of set equation.
5. Notion of relation between two sets and relation on some set, several examples in mathematics (belonginess, inclusion, divisibility, etc.) Matrix and graphical representation of relations. Operations with relations: union, intersection, complement, inverse relation. Matrices and graphs of these relations.
6. Main properties of relations given on a set: concepts of reflexive, symmetric, anti-symmetric, transitive relations, properties of their matrices and graphs.
7. Properties of relations given on a set. Concept of equivalence relation, concept of set partition. Theorem on connection between partitions and equivalence relations (factorization theorem).
8. Concepts of order relation, anti-reflexive relation and strict order. Concepts of ordered set, comparable and incomparable elements. Definitions of linear order and partial order, linearly ordered and partially ordered sets. Simplified form of a graph for any order relation (Hasse diagram), algorithm of its obtaining.
9. Concept of ordered set, comparable and incomparable elements. Minimal and maximal elements of ordered set. Theorem on existence of maximal and minimal elements in finite ordered set.
10. Functional relations and functions. Notions of function, its domain and range, examples. Definitions of injection, surjection and bijection, their properties. Inverse function, equality rule.
11. Countable and uncountable sets. Examples of countable sets. Proof that $\mathbb{R}$ is uncountable set.

12. Basic principles of counting. Equality rule, sum rule and product rule, examples of their application. Verification of the product rule by using of decision tree. Generalized sum and product rules.

13. Notion of a word in an alphabet, length of a word, the number of words having given length. Representation of binary words by binary trees.

14. Notion of alphabetic order and its extension to lexicographic order. Definition of lexicographic order, proof of transitivity for lexicographic relation.

15. Bijection between the set of binary words and the set of their decimal representations. Algorithm for computation of binary representation for decimal numbers based on manipulations with the powers of 2.

16. Definition of permutation, decision tree for selection of permutation. The number of all permutations having n elements. Notion of k-permutation of n-element set. The number of all k-permutations $P(n, k)$.

17. Definition of k-combination of n-element set. The number of all k-combinations $C(n, k)$. Properties of $C(n, k)$ numbers. Pascal triangle. Notion of binary words with prescribed distribution. The number of such words (prove that the number is $C(n, k)$).

18. Binomial Theorem (Newtonian Binomial), corollaries from it.

19. Partitions with given specification, their number, polynomial coefficients. Words with prescribed specification, their number. Polynomial Theorem.

20. Multisets, the number of multisets having prescribed size. Combinations with repetitions.

21. Inclusion-exclusion principle. Formulae for two, three, and four sets. Illustration of the formulae for $n = 2$ and $n = 3$ with a help of Venn diagram. Notion of derangement, application of inclusion-exclusion principle for counting the number of derangements.

22. Types of ordered partitions, ordered partitions with empty set allowed, their number. Examples. Ordered partitions with empty set forbidden, application of inclusion-exclusion principle for counting their number.

23. Unordered partitions with empty set forbidden, their number, second order Stirling numbers.

24. The number of all unordered partitions of a set having given size (Bell bumbers, the number of all equivalence relations on a set with given cardinality).

25. Functions of various types. The number of general functions, injections and bijections. Surjections, their number, connection with ordered partitions having non-empty parts. Strict and non-strict monotonic functions, their number.

26. First order linear recurrences with constant coefficients, algorithm of their solving. Example: towers of Hanoi.

27. Second order linear recurrences with constant coefficients. Homogenious second order equations, their solving algorithm, characteristic equation, cases of one root and two roots. Types of general and particular solutions. Examples: Fibonacci numbers and sparse words.

28. Second order linear recurrences with constant coefficients. Inhomogenious second order equations, their solving algorithm (reduction to homogeneous case). Two cases of unknown substitution. Types of general and particular solutions.

Section V. Problems for independent work.

Chapter 1. Sets and set operations.

1. Given a set $U = \{1, 2, \dots , 16\}$ and its subsets $A = \{x \in U : x \equiv 3 \pmod 4\}$, $B = \{x \in U : 11 < x \leqslant 15\}$, $C = \{x \in U : x$ is divided by 6 or by 8$\}$, $D = \{x \in U : x$ is divided by 9 or $x \equiv 1 \pmod 7\}$.
   Find sets $F = A \cup B$, $G = C \cap D$, $K = B \otimes C$, $L = (A - B) \cup (C - D)$, $2^A \cap 2^B$, $2^D - 2^B$, $A \times B$, $D \times C$, D^2. Calculate characteristic vectors for the sets F, G, K, L.

2. Solve set equation $A \cup X = X - B$, where A and B are subsets of some universe U. Determine necessary and sufficient conditions for existence of solution. Provided that necessary and sufficient conditions take place find the number of all solutions if $|U| = n$, $|A| = m$, $|B| = k$ $(m \leqslant n,\ k \leqslant n)$.

3. Solve set equation $A \cap X = B - X$, where A and B are subsets of some universe U. Determine necessary and sufficient conditions for existence of solution. Provided that necessary and sufficient conditions take place find the number of all solutions if $|U| = n$, $|A| = m$, $|B| = k$ $(m \leqslant n,\ k \leqslant n)$.

Chapter 2. Binary relations.

Given a universe $U = \{1, 2, \dots , 7\}$ and the following relations R_1 and R_2 on U: $R_1 = \{(x, y) \in U^2 : x \equiv y \pmod 3\}$, $R_2 = \{(x, y) \in U^2 : (x + 1)$ divides $(y + 1)\}$.

1. Construct matrices of R_1 and R_2. Draw graphs of R_1 and R_2.
2. Determine whether the relations R_1, R_2 are reflexive, symmetric, anti-symmetric, transitive.
3. Which of the relations R_1 and R_2 is equivalence relation? For the equivalence relation construct equivalence classes (factor set).
4. Which of the relations R_1 and R_2 is order relation? Is this order linear or partial? Draw Hasse diagram for the order relation. Find all maximal and minimal elements with respect to the order.
5. Construct graphs for the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \otimes R_2$, R_1^{-1}, R_2^{-1}, $R_1^{-1} \cup R_2$, $R_1 \cap R_2^{-1}$. Among them enumerate equivalence and order relations if they exist.

Chapter 3. Elements of combinatorics.

1. There are m books written by Shakespeare, n books of A.S. Pushkin and k books written by Leo Tolstoy. What is the number of ways to take in any order 2 books of Shakespeare, 3 books of Pushkin and 5 books of Tolstoy?

2. There are 22 students in academic group. Each of them likes football, basketball and volleyball. It's necessary to form from them 3 teams: football team must have 11 players, basketball team – 5 players, and volleyball team – 6 players (every student may be a member of only one team). Find the number of ways to partition the group by these 3 teams. (Use partitions with given specification)

3. There are 36 people living in a village. 21 of them grow potato, 18 – carrot, and 16 grow onion. 13 people grow potato and carrot, 12 – potato and onion, and 9 – carrot and onion. 5 people grow potato, carrot and onion. How many people living in the village do not grow each of these vegetables? Find the number of people growing precisely 2 kinds of the vegetables and precisely 1 kind of the vegetables. (Apply inclusion-exclusion method)

Chapter 4. Recurrence equations.

1. Solve first order linear recurrence $4x_n = 7x_{n-1} - 2$ under initial condition $x_0 = -1/2$.
2. Solve second order linear homogeneous recurrence

   $9x_n = 3x_{n-1} + 2x_{n-2}$ under initial conditions $x_0 = 2$, $x_1 = -3$.
3. Solve second order linear inhomogeneous recurrence

   $3x_n = -4x_{n-1} + 4x_{n-2} + 7$ under initial conditions $x_0 = -1$, $x_1 = 1$.

Sergei Vladimirovich **Sorochan**

FUNDAMENTALS OF DISCRETE MATHEMATICS

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